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Harmonic oscillations: A physical model to describe mechanical and electrical systems in harmonic motion

Abstract

Harmonic oscillations are mathematically described with a linear, ordinary differential equation of *2nd* order with constant coefficients. The article gives the phenomenological deduction of the describing differential equations of a simple mechanical and electrical system respectively and shows the technique to solve these equations. Starting with the distinction of real, complex or degenerated roots of the characteristic polynomial (eigenvalues of the differential equation), the description of a simple – non driven but damped oscillator showing periodic or a-periodic motions is done. In case of a periodically driven oscillator the specific motion is discussed.

1 Introduction – Physics of phenomena

Physics defines a harmonic oscillator when the intrinsic acting principle of any system is described by an linear <u>o</u>rdinary <u>d</u>ifferential <u>e</u>quation (ODE) of 2nd order with constant coefficients. So far the system is left to its own resources, this effectuates a type of motion exclusively describable with sine and cosine functions and showing a strict periodicity. In case of any external driving influence a superposition of this periodic motion and the response of the system to the external influences occur. Determining reason therefore is – in case of an oscillation mass in a mechanical system – a restoring force directly proportional to the actual displacement of the mass. The principal of a "restoring force – proportional to the actual displacement" can be generalised for any kind of "displacement" of a system[1-2]. In *Fig. 1* two systems are shown which are

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completely different in the physics of their acting principles. In case of the spring-mass system the displaceable system variable is the oscillating mass m, in case of the RCL serial circuit the "displaceable" system variable might be the electric charge q(t), stored in the capacitor, or the actual electric current I(t) inner the circuit. A detailed analysis of both systems – drawn back to Newton's axioms and Kirchhoff´s laws respectively makes clear that both systems are described mathematically by the same type of differential equation: an ordinary, linear differential equation of 2^{nd} order with constant coefficients. As it will become clear later in this paper it does not matter if a system like those is driven from external influences or not. Characteristics of these systems are not influenced from the existence of external influences or not – so far they do not change the fundamental parameters of the acting principle (like the constancy of the coefficients, the order of the differential equation or the linearity). The characteristic behaviour is exclusively determined by the existence of a "restoring force" proportional to the actual "displacement".

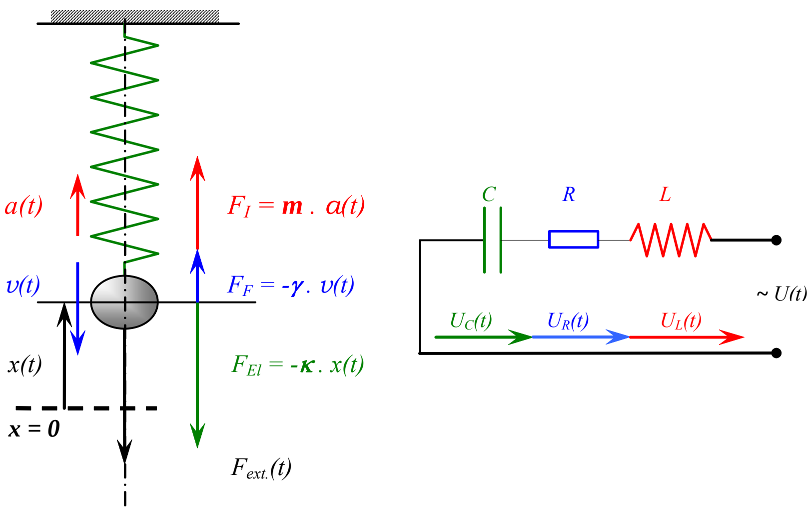

Fig. 1: Mechanic and electric oscillators:
spring-mass system and RCL circuit.

As shown in Tab. 1, for both systems figured out in Fig. 1 the physical analyses of the performing effects can easily be done. In the spring-mass system elastic force F_{El} – caused by the elastic properties of the spring, friction F_F – mainly caused from the viscous damping of the surrounding medium, the external force $F_{ext.}$ and inertia of the mass – F_I occurs. Notice that the elastic force acting on the mass is determined from the spring rate κ but inversely proportional to the actual displacement $x(t)$ (Hooke´s law[3]). A slow moving mass sees friction determined from the friction coefficient γ and in-

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versely proportional to the actual velocity (Stoke´s law[4]). Inertia is determined from mass m and proportional to the actual acceleration (Newton's law of inertia[5]). The RCL circuit shows voltage drops from the resistor *R*: U_R, the capacitor *C*: U_C and the inductivity *L*: U_L and the external voltage *U(t)*. The ohmic voltage drop U_R is determined from the resistance *R* and is proportional to the actual electric current *I(t)* (Ohm´s law[6]). The capacitive voltage drop U_C is determined from the capacitor *C* and is proportional to the actual electric charge *q(t)* – described by the integral of the current $\int I(t).dt$ over the charging time of the capacitor[7]. The inductive voltage drop UL is determined from the inductor *L* and is proportional to the first derivation versus time of actual electric current[8] *dI(t)/dt*. Equations (1.0-1) give an overview.

$$F_{ext.} = \kappa \cdot x(t) + \gamma \cdot \frac{dx(t)}{dt} + m \cdot \frac{d^2x(t)}{dt^2}$$

$$\frac{dU_{ext}}{dt}(t) = \frac{1}{C}.I(t) + R.\frac{d\dot{I}(t)}{dt} + L.\frac{d^2I(t)}{dt^2} \qquad (1.0\text{-}1)$$

Tab. 1: Balancing equations from Newton's law of inertia and Kirchhoff's mesh law

Newton's axiom of inertia	Kirchhoff´s mesh rule
$\sum \vec{F}_i = \sum \vec{F}_I = m \cdot a$	$\sum U_i = \sum U(t)$
$\vec{F}_{El} + \vec{F}_F + \vec{F}_{ext.} = \vec{F}_I = m \cdot \vec{a}$	$U_R + U_C + U_L = U(t)$
	$U(t) = R.I(t) + \frac{1}{C}.\int \dot{I}(t).dt + L.\frac{d\,I(t)}{dt} = R.\frac{d\,q(t)}{dt} + \frac{1}{C}.q(t) + L.\frac{d^2\,q(t)}{dt^2}$
	After derivation versus time:
$m \cdot \frac{d^2x(t)}{dt^2} = -\kappa \cdot x(t) - \gamma \cdot \frac{d\dot{x}(t)}{dt} + F_{ext}(t)$	$\frac{dU(t)}{dt} = \frac{1}{C} \cdot I(t) + R \cdot \frac{dI(t)}{dt} + L \cdot \frac{dI^2(t)}{dt}$

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After division by the coefficient of the highest order of derivation one obtains equations (1.0-2). Both equations (1.0-2) formally can become generalised to equation (1.0-4). The unusual notation of the coefficients as *2.h* and *k²* is chosen from practical reasons. This will become helpful in later calculations. For an easier notation derivations versus time are indicated as dotted symbols as introduced in (1.0-3). This leads to a common notation of time dependent ODE's as used in (1.0-4).

$$\frac{1}{m} \cdot F_{ext}(t) = \frac{d^2x(t)}{dt^2} + \frac{\gamma}{m} \cdot \frac{dx(t)}{dt} + \frac{\kappa}{m} \cdot x(t)$$

(1.0-2)

$$\frac{1}{L} \cdot U_{ext}(t) = \frac{d^2\,q(t)}{dt^2} + \frac{R}{L} \cdot \frac{d\,q(t)}{dt} + \frac{1}{L \cdot C} \cdot q(t) \quad \ldots.. \quad \left| \frac{d}{dt} \right.$$

$$\frac{1}{L} \cdot \frac{dU_{ext}(t)}{dt} = \frac{d^2I(t)}{dt^2} + \frac{R}{L} \cdot \frac{dI(t)}{dt} + \frac{1}{L.C} \cdot I(t)$$

$$\dot{f}(t) = \frac{df(t)}{dt} \; ; \quad \ddot{f}(t) = \frac{d^2 f(t)}{dt^2} \qquad (1.0\text{-}3)$$

$$\ddot{f}(t) + 2 \cdot h \cdot \dot{f}(t) + k^2 \cdot f(t) = g(t) \qquad (1.0\text{-}4)$$

In the next chapter a method is shown how to solve the describing ordinary, linear differential equations with constant coefficients. It should be noticed that the shown methods works even for ODE´s of higher order too. The solution is independent from the physical realisation of the oscillating system and can be generalised to every application

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2 Method to solve linear ODE's

2.1 Mathematical remarks

From principle the presented method is suitable to solve any linear ODE of an arbitrary order n with constant coefficients as given in (2.1-1). To indicate that the following remarks are valid for any dependence of an arbitrary variable (e.g.: time t, displacement x, Temperature T, …) the variable is indicated with q. Notice that equation (2.1-1) is:

- An ordinary differential equation (ODE) because the function $f(q)$ only depends from a single independent variable q.
- A linear differential equation because neither the function $f(q)$ nor any of its derivates $d(q)/dq_i$ is multiplied with the function $f(q)$ or any of its derivations.
- An ODE of order n – O(n) because the highest derivative is n.
- An ODE with constant coefficients because all coefficients ai are defined to be constants – and not functions of the variable q.
- Inhomogeneous because there is another non-zero function $g(q)$ on the right side of (2.1-0) – the so called source term, which is self-contained from $f(q)$ or any of its derivates $d(q)/dq_i$.

$$\frac{d^n f(q)}{dq^n} + a_{n-1} \cdot \frac{d^{n-1} f(q)}{dq^{n-1}} + ... + a_1 \cdot \frac{d f(q)}{dq} + a_0 \cdot f(q) = g(q) \qquad (2.1\text{-}1)$$

Mathematics teaches that any homogeneous ODE of order n is solved from exactly n linear independent eigenfunctions $f_i(q)$. These eigenfunctions define the basis of a vector space of the dimension n. Every arbitrary linear combination of all of these eigenfunctions as written in (2.1-2) is a solution of the homogeneous ODE $f^{(h)}(q)$.

$$f^{(h)}(q) = A_1 \cdot f_1(q) + A_2 \cdot f_2(q) + ... + A_{n-1} \cdot f_{n-1}(q) + A_n \cdot f_n(q) \qquad (2.1\text{-}2)$$

The general solution $f(q)$ of the inhomogeneous ODE (2.1-3) is the sum of the homogeneous solution $f^{(h)}(q)$ and a particular solution $f^{(p)}(q)$ which is constructed from the specifics of the source term. This will be explained later in chapter 2.2.

$$f(q) = f^{(h)}(q) + f^{(p)}(q) \tag{2.1-3}$$

From the principles of linearity the homogeneous solution $f^{(h)}(q)$ must be a function which reproduces itself by all derivations up to its n^{th} derivative. There is only one type of function which is known for a behaviour like this: an exponential function $e^{\lambda.q}$. Thus to find the homogeneous solution $f^{(h)}(q)$ the ansatz (2.1-4) is used.

$$f^{(h)}(q) = e^{\lambda \cdot q} \tag{2.1-4}$$

To apply (2.1-4) to the homogeneous version of differential equation (2.1-1) and to divide by $e^{\lambda.q}$ leads to the characteristic polynomial of the ODE for λ. The roots of this polynomial determine the so called eigenvalues of the ODE. They may be all or partly real or complex and they may be all or partly multiple zero. As it will be shown in detail in chapter 2.2 this influences significantly the structure of the solution of the homogeneous ODE. In case of a multiple zero a method must be applied to construct the required number of n linear independent eigenfunctions *{f1(q), ... fn(q)}* of the ODE. This will be discussed in chapter 2.2 in detail too.

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2.2 Method to solve homogeneous linear 2nd order ODE

As motivated in chapter 1 the describing differential equation of a harmonic oscillator is a linear, inhomogeneous ODE of *2nd* order. Thus equation (2.1-1) modifies to (2.2-1).

$$\frac{d^2\, f(q)}{dq^2} + a_1 \cdot \frac{d\, f(q)}{dq} + a_0 \cdot f(q) = g(q) \qquad (2.2\text{-}1)$$

As mentioned in 2.1 for this ODE exactly two linear independent solving functions exist. These two functions are found by the exponential ansatz (2.2-2). First and second derivative lead to $f'(q)$ and $f''(q)$. The expressions for $f'(q)$ and $f''(q)$ will be applied in (2.2-1). After division by $e^{\lambda \cdot q}$ (notice: $e^{\lambda \cdot q} \neq 0 \quad \forall\, q$) this leads to the characteristic polynomial of 2nd order in λ (2.2-3). Its roots of the polynomial λ_1 and λ_2 are given in (2.2-4). They determine the eigenvalues of the problem and the two linear independent eigenfunctions as well. And they define the basis of the two dimensional vector space of the general solution of the two dimensional ODE.

$$f(q) = e^{\lambda \cdot q} \quad \rightarrow \quad f'(q) = \lambda \cdot e^{\lambda \cdot q} \; ; \quad f''(q) = \lambda^2 \cdot e^{\lambda \cdot q} \qquad (2.2\text{-}2)$$

$$\lambda^2 + a_1 \cdot \lambda + a_0 = 0 \qquad (2.2\text{-}3)$$

$$\lambda_{1;2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^{\;2}}{4} - a_0} \qquad (2.2\text{-}4)$$

The two eigenvalues $\{\lambda_1;\, \lambda_2\}$ define the two eigenfunctions $\{f_1(q);\, f_2(q)\}$ - (2.2-5). Notice that these eigenfunctions are natural exponential functions. But in case of complex eigenvalues (negative discriminate) they describe periodic, sinusoidal and co-sinusoidal functions (Euler's identity and Moivre's formula respectively)[a]. Thus the

[a] Euler's identity or Moivre's formula: $e^{\pm i\alpha} = \cos\alpha \pm i \cdot \sin\alpha$

distinction in the three possible cases – real or complex or double zero eigenvalues – describes three different types of the solution. Real and double zero eigenvalues describe so called over damped solutions. They show an exponential decay. Complex eigenvalues cause so called under damped solutions. They show an oscillating behaviour. The complete solution $f^{(h)}(q)$ of the homogeneous differential equation is an arbitrary linear combination of $f_1(q)$ and $f_2(q)$ as given in (2.2-6).

$$f_1(q) = e^{\left(-\frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} - a_0}\right)\cdot q} \quad \text{and} \quad f_2(q) = e^{\left(-\frac{a_1}{2} - \sqrt{\frac{a_1^2}{4} - a_0}\right)\cdot q} \qquad (2.2\text{-}5)$$

$$f^{(h)}(q) = C_1 \cdot e^{\left(-\frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} - a_0}\right)\cdot q} + C_2 \cdot e^{-\left(-\frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} - a_0}\right)\cdot q} \qquad (2.2\text{-}6)$$

Constants of integration are specifies by setting initial conditions. The number of these conditions is strictly the same than the order of the differential equation – and the number of integration steps therefore. Notice that this number is exactly the same as the number of linear independent eigenfunctions. Because of an 2^{nd} order problem the two numbers of integration C_1 and C_2 will be specified by 2 initial conditions.

2.2.1 Real but different eigenvalues λ_1 and λ_2

The solution of the ODE is as given in (2.2-6) – and repeated here.

$$f^{(h)}(q) = C_1 \cdot e^{\left(-\frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} - a_0}\right)\cdot q} + C_2 \cdot e^{-\left(-\frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} - a_0}\right)\cdot q} \qquad (2.2\text{-}6)$$

Because of a 2^{nd} order problem the two numbers of integration C_1 and C_2 will be specified by 2 initial conditions.

2.2.2 Real but equal eigenvalues $\lambda_1 = \lambda_2$

In the particular case that $a_1^2/4 - a_0 = 0$ identical values for λ_1 and λ_2 are given from equation (2.2-4). Thus only one eigenfunctions $f^{(h)}(q)$ can be found for the moment (2.2-7). This is in conflict with the requirement of linear algebra that two linear independent eigenfunctions must exist.

$$f^{(h)}(q) = C \cdot e^{-\frac{a_1}{2} q} \tag{2.2-7}$$

To fulfil this requirement the eigenfunction (2.2-7) is modified by generalising C to a function $C(q)$. With this a new eigenfunction (2.2-8) can be given.

$$\begin{aligned}
\hat{f}^{(h)}(q) &= C(q) \cdot e^{-\frac{a_1}{2} q} \\
\frac{d\hat{f}^{(h)}(q)}{dq} &= \frac{dC(q)}{dq} \cdot e^{-\frac{a_1}{2} q} - C(q) \cdot \frac{a_1}{2} \cdot e^{-\frac{a_1}{2} q} \\
\frac{d^2\hat{f}^{(h)}(q)}{dq^2} &= \frac{d^2C(q)}{dq^2} \cdot e^{-\frac{a_1}{2} q} - 2 \cdot \frac{a_1}{2} \cdot \frac{dC(q)}{dq} \cdot e^{-\frac{a_1}{2} q} + C(q) \cdot \left(\frac{a_1}{2}\right)^2 \cdot e^{-\frac{a_1}{2} q}
\end{aligned} \tag{2.2-8}$$

Applying (2.2-8) in (2.2-1) and division by the exponential term (notice: it is never equal to zero!) gives a simple differential equation for $C(q)$: $C''(q) = 0$. Notice that the second order of this differential equation comes from the fact that (2.2-1) id from 2nd order too. To integrate to times gives the function $C(q)$ - (2.2-9).

$$C(q) = C_2 \cdot q + C_1 \tag{2.2-9}$$

The two linear independent eigenfunctions ant the corresponding eigenvalues are given in (2.2-10). The solution of the homogeneous differential equation with a double-zero of the characteristic polynomial is (2.2-11)

$$f_1^{(h)}(q) = C_1 \cdot e^{-\frac{a_1}{2} \cdot q} \quad \rightarrow \quad \lambda_1 = C_1$$
$$f_2^{(h)}(q) = C_2 \cdot q \cdot e^{-\frac{a_1}{2} \cdot q} \quad \rightarrow \quad \lambda_2 = C_2 \cdot q \tag{2.2-10}$$

$$f^{(h)}(q) = \left(C_1 + C_2 \cdot q\right) \cdot e^{-\frac{a_1}{2} \cdot q} \tag{2.2-11}$$

With the rule of de l´ Hospital[9] it can be easily shown that $f^{(h)}(q)$ vanishes for infinite q when a_1 is a positive number (what is ensured for physical systems).

This method can be generalised for multiple zeros. Let λ_i be an eigenvalue of an-ode of order n and a multiple zero of the order j. With (2.2-8) a number of j linear independent eigenfunctions can be found as given in (2.2-12).

$$f_{1;i}^{(h)}(q) = C_{1;i} \cdot e^{-\frac{a_1}{2} \cdot q} \quad \rightarrow \quad \lambda_{1;i} = C_{1;i}$$
$$f_{2;i}^{(h)}(q) = C_{2;i} \cdot q \cdot e^{-\frac{a_1}{2} \cdot q} \quad \rightarrow \quad \lambda_{2;i} = C_{2;i} \cdot q \tag{2.2-12}$$
$$f_{j;i}^{(h)}(q) = C_{j;i} \cdot q^{j-1} \cdot e^{-\frac{a_1}{2} \cdot q} \quad \rightarrow \quad \lambda_{j;i} = C_{2;i} \cdot q^{j-1}$$

Because of a *2nd* order problem the two numbers of integration C_1 and C_2 will be specified by 2 initial conditions.

2.2.3 Complex eigenvalues λ_1 and λ_2

In the characteristic polynomial (2.2-4) the discriminate $a_1{}^2/4 - a_0$ is negative. From complex algebra one knows that a complex argument of exponential function represents sinusoidal and co-sinusoidal functions respectively. Thus one defines the absolute value $| a_1{}^2/4 - a_0 |$ as an angular frequency ω (eigen – angular – frequency) and transforms as shown in (2.2-13). Equation (2.2-6) transforms in (2.2-14). Using Euler's identity[b] and a re-definition of the integration constants C_1 and C_2 equation (2.2-14) can be transformed in (2.2-15)

$$\sqrt{\frac{a_1^{\ 2}}{4} - a_0} = \sqrt{(-1) \cdot \left| \frac{a_1^{\ 2}}{4} - a_0 \right|} = \sqrt{(-1) \cdot \omega^2} = i \cdot \omega \tag{2.2-13}$$

$$f^{(h)}(q) = C_1 \cdot e^{\left(-\frac{a_1}{2} + i \cdot \omega\right) \cdot q} + C_2 \cdot e^{\left(-\frac{a_1}{2} - i \cdot \omega\right) \cdot q} \tag{2.2-14}$$

With the following steps in calculation (2.2-15) is obtained.

$$f_1^{(h)}(q) = C_1 \cdot e^{\left(-\frac{a_1}{2} + i \cdot \omega\right) \cdot q} = \left(C_1 \cdot e^{-\frac{a_1}{2} \cdot q} \right) \cdot \left[\cos(\omega.q) + i \cdot \sin(\omega.q) \right]$$

$$f_2^{(h)}(q) = C_2 \cdot e^{\left(-\frac{a_1}{2} - i \cdot \omega\right) \cdot q} = \left(C_2 \cdot e^{-\frac{a_1}{2} \cdot q} \right) \cdot \left[\cos(\omega.q) - i \cdot \sin(\omega.q) \right]$$

$$\hat{f}_1^{(h)}(q) = \frac{1}{2} \cdot \left[f_1^{(h)}(q) + f_2^{(h)}(q) \right] = \frac{C_1 + C_2}{2} \cdot e^{-\frac{a_1}{2} \cdot q} \cdot \cos(\omega.q)$$

$$\hat{f}_2^{(h)}(q) = \frac{1}{2i} \cdot \left[f_1^{(h)}(q) - f_2^{(h)}(q) \right] = \frac{C_1 - C_2}{2i} \cdot e^{-\frac{a_1}{2} \cdot q} \cdot \sin(\omega.q)$$

$$f^{(h)}(q) = e^{-\frac{a_1}{2} \cdot q} \cdot \left[C_1 \cdot \sin(\omega \cdot q) + C_2 \cdot \cos(\omega \cdot q) \right] \tag{2.2-15}$$

Because of a *2nd* order problem the two numbers of integration C_1 and C_2 will be specified by 2 initial conditions.

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[b] Euler's identity or Moivre's formula: $e^{\pm i\alpha} = \cos\alpha \pm i \cdot \sin\alpha$

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2.3 Method to find particular solutions for inhomogeneous linear 2nd order ODE

Keep in mind that the solution of the homogeneous differential equation $f^{(h)}(q)$ fulfils the homogeneous requirement $f''(q) + a_1.f'(q) + a_0.f(q) = 0$. Thus the particular solution $f^{(p)}(q)$, which has to be added to the homogeneous solution to fulfil the inhomogeneous ODE $f''(q) + a_1.f'(q) + a_0.f(q) = g(q)$ has to be strictly adjusted to the source term $g(q)$. An ansatz for $f^{(p)}(q)$ must be found which transforms $f^{(p)}(q)$, in a way that $g(q)$ arises when the differential operator of the ODE is applied to $f^{(p)}(q)$. For this no method exists which works sufficiently for every arbitrary $g(q)$. But from principle one can say that the ansatz for $g(q)$ must be from the same function space than $g(q)$ itself. This means that a polynomial source term requires a polynomial ansatz, an exponential source term requires an exponential ansatz, a periodic oscillating $g(q)$ requires a superposition of sinusoidal and co-sinusoidal functions of the same periodicity than $g(q)$.

2.3.1 Let be $f(q) = \sum_{i=0}^{n} b_i \cdot q^i$ a polynomial of order n

From the general remarks of chapter 2.3 a polynomial ansatz (2.3-1) is chosen.

$$f^{(p)}(q) = \sum_{i=0}^{n} d_i \cdot q^i \qquad (2.3\text{-}1)$$

Example: $f''(q) + 4 \cdot f(q) = q^2$

The source term $g(q)$ is q^2. The solution for the homogeneous equation is: $f^{(h)}(q) = C_1.e^{i.2q} + C_2.e^{-i.2q}$. To construct $g(q)$ when applying the differential operator of the ODE: $[d^2/dq^2 + 4]$ to $f^{(p)}(q)$ a polynomial ansatz of order 2 is chosen: $f^{(p)}(q) = d_2.q^2 + d_1.q + d_0$. With the derivatives: $f^{(p)\prime} = 2.d_2.q + d_1$ and $f^{(p)\prime\prime} = 2.d_2$ the ODE becomes

$$2.d_2 + 4.(d_2.q^2 + d_1.q + d_0) = q^2$$

By comparison of the coefficients one finds:

$$q^2: 4.d_2 = 1;\ q: 4.d_1 = 0;\ q^0: 4.d_0 + 2.d_2 = 1:$$

$\rightarrow d_2 = {}^1\!/_4;\ d_1 = 0;\ d_0 = -{}^1\!/_8$

$f(q) = [C_1.e^{i.2q} + C_2.e^{-i.2q}] + {}^1\!/_4.q^2 - {}^1\!/_8.$

2.3.2 Let be $f(q) = m \cdot e^{n.q}$ an exponential function

From the general remarks of chapter 2.3 from principle an exponential ansatz (2.3-b) is chosen. But it is to proof that the exponent *n.q* is not exponent of the homogeneous solution $f^{(h)}(q)$. In case if this only a linear dependent variation of the solution of the homogeneous ODE would be constructed – equalising the inhomogeneous problem to zero!

$$f^{(p)}(q) = d \cdot e^{n \cdot q} \qquad (2.3\text{-}2)$$

To avoid the problem one generalises (2.3-2a) to (2.3-2b). The exponent *k* is the lowest possible exponent of *q* which is higher than the highest exponent of *q* in the multiple zero of highest order.

$$f^{(p)}(q) = d \cdot q^k \cdot e^{n \cdot q} \qquad (2.3\text{-}3)$$

<u>Example:</u> $f''(q) + 2 \cdot f'(q) + f(q) = 3 \cdot e^{-q}$

The characteristic polynomial of the homogeneous solution has a double zero. The homogenous solution is: $f^{(h)}(q) = (C_1 + C_2.q).e^{-q}$. The lowest possible *k*-exponent of *q* higher than the highest exponent of the double zero is: $k = 2$. Thus $f^{(p)}(q) = d.q^2.e^{-q}$. With its derivatives $f'^{(p)}(q) = d.e^{-q}.(2.q - q^2)$ and $f''^{(p)}(q) = d.e^{-q}.(q^2 - 4.t + 2)$ the ODE becomes:

$$d.e^{-q}.(q^2 - 4.q + 2) + 2.d.e^{-q}.(2.q - q^2) + d.q^2.e^{-q} = 3.e^{-q}$$

By comparison of the coefficients one finds:

q^2: no conclusion; q^1: no conclusion; q^0: $2.d = 3;$

$\rightarrow d = {}^3\!/_2$

$f(q) = (C_1 + C_2.q + {}^3\!/_2.q^2).e^{-q}$

2.3.3 Let be $g(q) = g_0 \cdot \sin(\Omega \cdot q)$ a sinusoidal function

From the general remarks of chapter 2.3 from principle an exponential ansatz (2.3-4) is chosen. The parameter Ω is called excitation frequency.

$$f^{(p)}(q) = D_1 \cdot \sin(\Omega \cdot q) + D_2 \cdot \cos(\Omega \cdot q) \tag{2.3-4}$$

<u>Example:</u> $f''(q) + a_1 \cdot f'(q) + a_0 \cdot f(q) = g_0 \cdot \sin(\Omega \cdot q)$

The solution of the homogeneous ODE is given in (2.2-15). With the derivatives

$$\begin{aligned} f'^{(p)}(q) &= \Omega \cdot [D_1 \cdot \cos(\Omega \cdot q) - D_2 \cdot \sin(\Omega \cdot q)] \\ f''^{(p)}(q) &= -\Omega^2 \cdot [D_1 \cdot \sin(\Omega \cdot q) + D_2 \cdot \cos(\Omega \cdot q)] \end{aligned} \tag{2.3-5}$$

one obtains:

$$\begin{aligned} & -\Omega^2 \cdot [D_1 \cdot \sin(\Omega \cdot t) + D_2 \cdot \cos(\Omega \cdot t)] + \\ & \quad + 2 \cdot h \cdot \Omega \cdot [D_1 \cdot \cos(\Omega \cdot t) - D_2 \cdot \sin(\Omega \cdot t)] + \\ & \quad\quad + k^2 \cdot D_1 \cdot \sin(\Omega \cdot t) + D_2 \cdot \cos(\Omega \cdot t) \qquad = \quad g_0 \cdot \sin(\Omega \cdot t) \end{aligned} \tag{2.3-6}$$

By comparison of the coefficients one finds:

$$\begin{aligned} \sin(\Omega \cdot q): &\qquad D_1 \cdot (k^2 - \Omega^2) + D_2 \cdot (-2 \cdot h \cdot \Omega) = g_0 \\ \cos(\Omega \cdot q): &\qquad D_1 \cdot (2 \cdot h \cdot \Omega) + D_2 \cdot (k^2 - \Omega^2) = 0 \end{aligned} \tag{2.3-7}$$

This is a linear system of equations with the variables D_1 and D_2. A possibility to solve linear system of equations is to apply Cramer's rule[10]. The solution is expressed in determinants of the coefficients *K* and both variable D_1 and D_2.

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$$Det_K = \begin{vmatrix} k^2 - \Omega^2 & -2.h.\Omega \\ 2.h.\Omega & k^2 - \Omega^2 \end{vmatrix}$$

$$Det_{D1} = \begin{vmatrix} g_0 & -2.h.\Omega \\ 0 & k^2 - \Omega^2 \end{vmatrix} \qquad (2.3\text{-}8)$$

$$Det_{D2} = -\begin{vmatrix} k^2 - \Omega^2 & g_0 \\ 2.h.\Omega & 0 \end{vmatrix}$$

With this the coefficients D_1 and D_2 can be calculated.

$$D_1 = \frac{Det_{D1}}{Det_K} = \frac{(k^2 - \Omega^2).g_0}{\left(k^2 - \Omega^2\right)^2 + 4 \cdot h^2 \cdot \Omega^2}$$

$$(2.3\text{-}9)$$

$$D_2 = \frac{Det_{D2}}{Det_K} = \frac{2.h.\Omega.g_0}{\left(k^2 - \Omega^2\right)^2 + 4 \cdot h^2 \cdot \Omega^2}$$

$$f^{(h)}(q) = e^{-\frac{a_1}{2}\cdot q} \cdot \left[C_1 \cdot \sin(\omega \cdot q) + C_2 \cdot \cos(\omega \cdot q)\right] \qquad (2.3\text{-}10)$$
$$f^{(p)}(q) = D_1 \cdot \sin(\Omega \cdot q) + D_2 \cdot \cos(\Omega \cdot q)$$

$$f(q) = e^{-\frac{a_1}{2}\cdot q} \cdot \left[C_1 \cdot \sin(\omega \cdot q) + C_2 \cdot \cos(\omega \cdot q)\right] + \qquad (2.3\text{-}11)$$
$$+ \left[D_1 \cdot \sin(\Omega \cdot q) + D_2 \cdot \cos(\Omega \cdot q)\right]$$

The phenomenon is discussed in detail in the next chapter.

3 Specification to harmonic oscillations

From introduction dislocations of a mass, of an electric charge and of electric current are known. They became described by the following equations (3.0-1).

$$\frac{d^2x(t)}{dt^2}+\frac{\gamma}{m}.\frac{d\dot{x}(t)}{dt}+\frac{\kappa}{m}.x(t) \;=\; \frac{1}{m}.F_{ext}(t)$$

$$\frac{d^2\,q(t)}{dt^2}+\frac{R}{L}.\frac{dq(t)}{dt}+\frac{1}{L\cdot C}\cdot q(t) \;=\; \frac{1}{L}.U_{ext}(t) \qquad (3.0\text{-}1)$$

$$\frac{d^2I(t)}{dt^2}+\frac{R}{L}.\frac{d\dot{I}(t)}{dt}+\frac{1}{L\cdot C}\cdot I(t) \;=\; \frac{1}{L}\cdot\frac{dU_{ext}(t)}{dt}$$

These examples for the mathematical description of a physical system and lots of others can formally be formulated like (3.0-2). To discuss the phenomenon in general a describing function *f(t)* is used. Time dependence of the physical quantities is assumed. This is from phenomenological reasons only. As it was discussed in chapter 2.1 no restrictions in the variable of the phenomena exists. But one can imagine that displacement, temperature and other variable might be used to describe the dependency of a physical effect. So formally oscillation phenomena are formulated as done in (3.0-2).

$$\ddot{f}(t)+2\cdot h\cdot\dot{f}(t)+k^2\cdot f(t)=g(t) \qquad (3.0\text{-}2)$$

Equation (3.0-2) is solved in three steps:

- Solution of the homogenous differential equation with an exponential ansatz $f^{(h)}(t) = e^{\lambda.t}$. Notice that the number of different eigenvalues λ_i ad of linear independent eigenfunctions $f_i^{(h)}(t)$ is determined from the order of the ODE: here one has to find two linear independent eigenfunctions.
- Solution of the inhomogeneous differential equation with an ansatz dependent from the source term *g(t)*.
- Identification of the constants of integration from two initial conditions. Because of a 2^{nd} order problem the two numbers of integration C_1 and C_2 will be specified by 2 initial conditions.

3.1 Free oscillation – non driven oscillator

The system is described by the homogenous equation (3.1-1). As discussed earlier an exponential ansatz $f^{(h)}(t) = e^{\lambda.t}$ is made. After division by $e^{\lambda \cdot t}$ (notice: $e^{\lambda t} \neq 0 \quad \forall\, t$) this leads to the characteristic polynomial of 2nd order in λ (3.1-2). Its roots λ_1 and λ_2 are given in (3.1-3). They determine the eigenvalues of the problem and the two linear independent eigenfunctions as well. And they define the basis of the two dimensional vector space of the general solution of the two dimensional ODE. The determinants of the acting physical principle specify the two coefficients h and k^2. Thus the distinction in real or complex or double zero eigenvalues is determined by these determinants.

$$\ddot{f}^{(h)}(t) + 2h \cdot \dot{f}^{(h)}(t) + k^2 \cdot f^{(h)}(t) = 0 \qquad (3.1\text{-}1)$$

$$\lambda^2 + 2 \cdot h \cdot \lambda + k^2 = 0 \qquad (3.1\text{-}2)$$

$$\lambda_{1;2} = -h \pm \sqrt{h^2 - k^2} \qquad (3.1\text{-}3)$$

The two eigenvalues $\{\lambda_1;\ \lambda_2\}$ define the two eigenfunctions $\{x_1(t);\ x_2(t)\}$ - (3.1-4). Notice that these eigenfunctions are natural exponential functions. But in case of complex (negative discriminate $h^2 - k^2$) eigenvalues they describe periodic, sinusoidal and co-sinusoidal functions (Euler's identity and Moivre's formula respectively)[c]. Thus the distinction in the three possible cases – real or complex or double zero eigenvalues – describes three types of motion, all realised in nature. Real and double zero eigenvalues cause a-periodic motions, complex eigenvalues oscillating motions. From principle but simplifying one can say: whenever friction dominates the elastic restoring force, a-periodic motion occurs. Elastic dominated systems move periodically. Substituting like (3.1-5): $h^2 - k^2 = p^2$ transforms eigenfunctions from (3.1-4) to (3.1-6)

$$f_1(t) = e^{\left(-h+\sqrt{h^2-k^2}\right)\cdot t} \quad ; \quad f_2(t) = e^{-\left(h+\sqrt{h^2-k^2}\right)\cdot t} \qquad (3.1\text{-}4)$$

$$p^2 = h^2 - k^2 \qquad (3.1\text{-}5)$$

$$f_1(t) = e^{(-h+p)\cdot t} \quad ; \quad f_2(t) = e^{-(h+p)\cdot t} \qquad (3.1\text{-}6)$$

---

c Euler's identity or Moivre's formula: $e^{\pm i\alpha} = \cos\alpha \pm i \cdot \sin\alpha$

3.1.1 Over damped oscillator: *$h^2 - k^2 > 0$* A-periodic motion

The over damped oscillator is caused from physical conditions leading to real roots with a discriminate $h^2 - k^2 > 0$. The homogeneous differential equation (3.1-1) has the solution (3.1-7). Over damped motion with $h^2 - k^2 > 0$ becomes realised when friction is dominant compared to the restoring effect of the system. Because both values of h and k are positive from physical reasons, from any displacement the system trends back to zero what defines its rest position. The behaviour of an over damped oscillator is shown in Fig. 2. Notice that specific initial conditions are defined. Keep in mind that one is absolutely free in the choice of these initials! The only constraint is that the initials define a real system. A typical choice is done in (3.1-8) where values refer to an initial time $t_0 = 0$. To insert these conditions in (3.1-7) gives the constants of integration (3.1-9). The solution is given in (3.1-10) and shown in Fig. 2.

$$f^{(h)}(t) = C_1 \cdot e^{(-h+p)t} + C_2 \cdot e^{-(h+p)t} \tag{3.1-7}$$

$$f(t_0 = 0) = f_0 , \quad \dot{f}(t_0 = 0) = \dot{f}_0 \tag{3.1-8}$$

$$C_1 = \frac{(p+h) \cdot f_0 + \dot{f}_0}{2 \cdot p} ; \qquad C_2 = \frac{(p-h) \cdot f_0 - \dot{f}_0}{2 \cdot p} \tag{3.1-9}$$

$$f^{(h)}(t) = \frac{(p+h) \cdot f_0 + \dot{f}_0}{2 \cdot p} \cdot e^{(-h+p)t} + \frac{(p-h) \cdot f_0 - \dot{f}_0}{2 \cdot p} \cdot e^{-(h+p)t} \tag{3.1-10}$$

System:

Oscillator

R [Ω]	*C^{-1} [F^{-1}]*	*L [H]*
γ [Nsm^{-1}]	*κ [Nm^{-1}]*	*m [kg]*
5	1	1

Initial conditions

f_0 [A]	*$f_{;t0}$ [As^{-1}]*
f_0 [m]	*$f_{;t0}$ [ms^{-1}]*
0,5	1

Parameters, constants of integration

2h	*k²*	*p*
5	1	2,291287847
h²	*k²*	
6,25	1	
C_1	C_2	
0,740990253	-0,24099025	
A	*φ*	

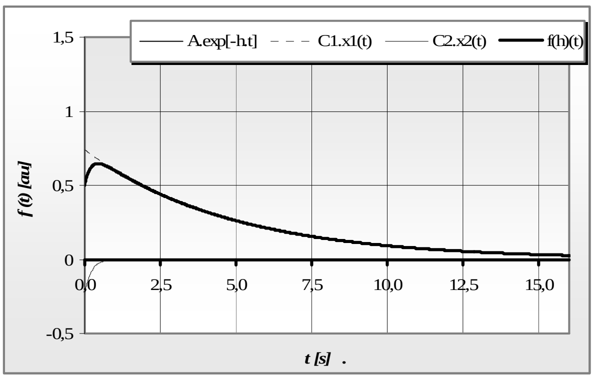

Fig. 2: Over damped oscillator

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3.1.2 Critically damped oscillator: *h² - k² = 0* A-periodic motion

The over damped oscillator is caused from physical conditions leading to real roots with a discriminate $h^2 - k^2 = 0$. The homogeneous differential equation (3.1-1) has the solution (3.1-11). Critically damped motion with $h^2 - k^2 = 0$ becomes realised when friction exactly equalises the restoring effect of the system. Because both values of h and k are positive from physical reasons, from any displacement the system trends back to zero what defines its rest position. The behaviour of an over damped oscillator is shown in Fig. 3. Notice that specific initial conditions are defined. Keep in mind that one is absolutely free in the choice of these initials! The only constraint is that the initials define a real system. A typical choice is done in (3.1-11) where values refer to an initial time $t_0 = 0$. To insert these conditions in (3.1-10) gives the constants of integration (3.1-12). The solution (3.1-13) is shown in Fig. 3. A comparison of the critically damped oscillator with the over damped oscillator shows that critical damping allows the system to trend back in its rest position in a minimum of time. This is often desired in damped systems like doors, cars, or electric circuits when switching off an inductor.

$$f^{(h)}(t) = (C_1 + C_2 t) \cdot e^{-h \cdot t} \qquad (3.1\text{-}11)$$

$$f(t_0 = 0) = f_0, \quad \dot{f}(t_0 = 0) = \dot{f}_0 \qquad (3.1\text{-}11)$$

$$C_1 = f_{0;} \qquad C_2 = \dot{f}_0 + h \cdot f_0 \qquad (3.1\text{-}12)$$

$$f^{(h)}(t) = \left[f_0 + \left(\dot{f}_0 + h \,.\, f_0 \right) .\, t \right] \cdot e^{-h \cdot t} \qquad (3.1\text{-}13)$$

System:

Oscillator

R [Ω]	*C⁻¹ [F⁻¹]*	*L [H]*
γ [Nsm⁻¹]	*κ [Nm⁻¹]*	*m [kg]*
2	1	1

Initial conditions

f_0 [A]	$f_{;t0}$ [As⁻¹]
f_0 [m]	$f_{;t0}$ [ms⁻¹]
0,5	1

Parameters, constants of integration

2h	*k²*	*p*
2	1	0
h²	*k²*	
1	1	
C_1	C_2	
0,5	1,5	
A	*φ*	

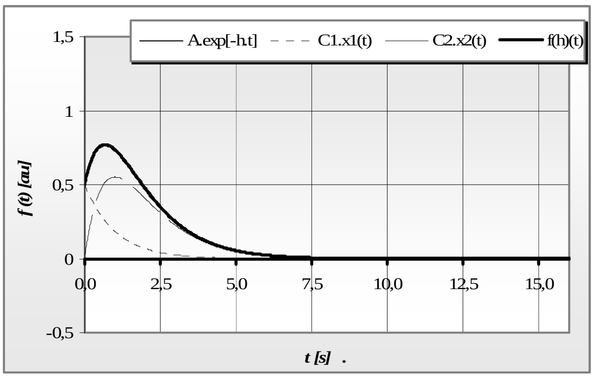

Fig. 3: Critically damped oscillator

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3.1.3 Under damped oscillator: $h^2 - k^2 < 0$ Periodic motion

The under damped oscillator is caused from physical conditions leading to real roots with a discriminate $h^2 - k^2 < 0$. The homogeneous differential equation (3.1-1) has the solution (3.1-14). The parameter ω is called eigen angular frequency. Under damped motion with $h^2 - k^2 < 0$ becomes realised when the restoring effect of the system dominates friction. The system is oscillation. Nevertheless: When friction happens from any displacement the system trends back to zero what defines its rest position. Once more the reason is that both values of h and k are positive from physical reasons! Therefore a both oscillation and damping occur. This causes decreasing amplitudes of motion. The behaviour of an under damped oscillator is shown in Fig. 4. Notice that specific initial conditions are defined. Keep in mind that one is absolutely free in the choice of these initials! The only constraint is that the initials define a real system. A typical choice is done in (3.1-15) where values refer to an initial time $t_0 = 0$. To insert these conditions in (3.1-14) gives the constants of integration (3.1-16).

$$f^{(h)}(t) = e^{-h \cdot t} \cdot \left[C_1 \cdot \sin(\omega.t) + C_2 \cdot \cos(\omega.t)\right] \qquad (3.1\text{-}14)$$

$$f(t_0 = 0) = f_0, \quad \dot{f}(t_0 = 0) = \dot{f}_0 \qquad (3.1\text{-}15)$$

$$C_1 = \frac{\dot{f}_0 + h \cdot f_{0;}}{\omega}; \qquad C_2 = f_0 \qquad (3.1\text{-}16$$

From practical reasons it makes sense to substitute for C_1 and C_2 the following expressions (3.1-17). Quadratic addition and division of C_2/C_1 respectively gives (3.1-18). By use of the well known identity: $\sin(\alpha \pm \beta) = \sin\alpha \,.\, \sin\beta \pm \cos\alpha \,.\, \cos\beta$ another representation of the homogeneous solution of the differential equation of the oscillating system can be written (3.1-19). Results are shown in Fig. 4.

$$C_1 = A \cdot \cos\varphi; \qquad C_2 = A \cdot \sin\varphi \qquad (3.1\text{-}17)$$

$$A = \sqrt{C_1^{\,2} + C_2^{\,2}}; \qquad \varphi = atg\,\tfrac{C_2}{C_1} \qquad (3.1\text{-}18)$$

$$f^{(h)}(t) = A \cdot e^{-h \cdot t} \cdot \sin(\omega \cdot t + \varphi) \qquad (3.1\text{-}19)$$

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System:

Oscillator

R [Ω]	C^{-1} [F^{-1}]	L [H]
γ [Nsm^{-1}]	κ [Nm^{-1}]	m [kg]
0,5	1	1

Initial conditions

f_0 [A]	$f_{;t0}$ [As^{-1}]
f_0 [m]	$f_{;t0}$ [ms^{-1}]
0,5	1

Parameters, constants of integration

$2h$	k^2	w
0,5	1	0,968245837
h^2	k^2	
0,0625	1	
C_1	C_2	
1,161895004	0,5	
A	φ	
1,264911064	0,406377781	

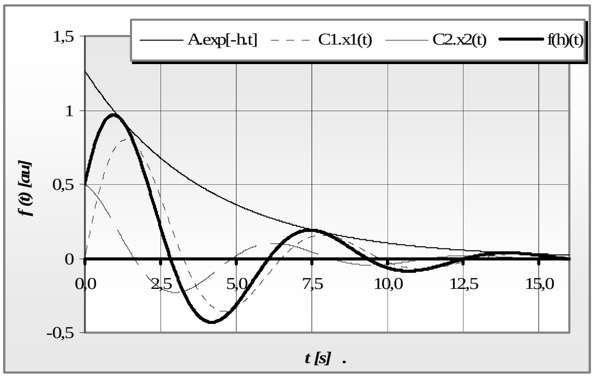

Fig. 4: Under damped – oscillation oscillator

3.2 External driven oscillator: source term

As it is known from chapters 2.3 and 2.3.3 formula (3.2-1) is solution of the describing differential equation of a harmonic oscillator with a sinusoidal source term. The parameter Ω is called excitation angular frequency. It is strictly to distinguish from the eigen angular frequency ω of the oscillation system. The mathematical description of the driven oscillator (solution of the inhomogeneous problem) is known from (2.3-11).

$$f(t) = e^{-h\cdot t} \cdot \left[C_1 \sin(\omega \cdot t) + C_2 \cos(\omega \cdot t)\right] + \left[D_1 \sin(\Omega \cdot t) + D_2 \cos(\Omega \cdot t)\right]$$

$$D_1 = \frac{(k^2 - \Omega^2).g_0}{\left(k^2 - \Omega^2\right)^2 + 4 \cdot h^2 \cdot \Omega^2} \tag{3.2-1}$$

$$D_2 = \frac{2.h.\Omega.g_0}{\left(k^2 - \Omega^2\right)^2 + 4 \cdot h^2 \cdot \Omega^2}$$

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From practical reasons it makes sense to substitute for D_1 and D_2 the following expressions (3.2-2). Quadratic addition and division of D_2/D_1 respectively gives (3.2-3). By use of the well known identity:

$$\sin(\alpha \pm \beta) = \sin\alpha \cdot \sin\beta \pm \cos\alpha \cdot \cos\beta$$

a representation of the inhomogeneous solution of the differential equation analogue to the solution of the homogeneous solution of the oscillating system can be written. Here one has to understand that δ is to calculate as the negative inverse tangens (D_2/D_1). The reason is that the oscillation of the physical system is the answer to the external influences from the source term. Thus it shows a negative phase shift, because it arises later than its direct cause (3.2-4).

$$D_1 = B \cdot \cos\delta \,; \qquad D_2 = B \cdot \sin\delta \tag{3.2-2}$$

$$B = \sqrt{D_1^{\,2} + D_2^{\,2}} \tag{3.2-3}$$

$$\delta \overset{!}{=} -atg\left({D_2}/{D_1} \right) \overset{!}{=} -\,atg\,\frac{2.h.\Omega}{k^2 - \Omega^2} \tag{3.2-4}$$

The superposition of the homogeneous and the inhomogeneous solution is given with equation (3.2-5). Thus two equivalent mathematical descriptions of a periodically driven harmonic oscillator can be given: (3.2-4).

$$\begin{aligned} f(t) &= e^{-h\cdot t} \cdot [A \cdot \sin(\omega \cdot t + \varphi)] + [B \cdot \sin(\Omega \cdot t + \delta)] \\ f(t) &= e^{-h\cdot t} \cdot [C_1 \sin(\omega \cdot t) + C_2 \cos(\omega \cdot t)] + [D_1 \sin(\Omega \cdot t) + D_2 \cos(\Omega \cdot t)] \end{aligned} \tag{3.2-5}$$

It is clear from (3.2-5) that the homogenous part of the solution vanishes with ongoing time. Reason therefore comes from friction which causes the damping of the homogenous solution. After a sufficiently long period of time the oscillation of the driven oscillator only pictures the sours term.

Notice that the parameters of the particular solution $\{B, \delta\}$ and $\{D_1, D_2\}$ respectively are caused from the source term and the system parameters of the oscillator. They do not depend from any initial condition! Nevertheless they influence the constants of integration $\{A, \varphi\}$ and $\{C_1, C_2\}$ respectively when initial conditions (3.2-7) are formulated. Constants of integration transform from those as determined for an undriven oscillator (3.1-16) to a driven oscillator (3.2-6).

$$f(t_0 = 0) = f_0 \,, \quad \dot{f}(t_0 = 0) = \dot{f}_0 \tag{3.2-6}$$

$$C_1 = \frac{\dot{f}_0 + h \cdot (f_0 - D_2) - \Omega \cdot D_1}{\omega} \,; \qquad C_2 = f_0 - D_2 \tag{3.2-7}$$

For example the oscillator with identical physical system parameters and with identical initial conditions is used than shown in Fig. 4. Now it is driven with an external periodic source as indicated in Tab. 2. This transforms the constants of integration. In Tab. 2 both types of representation: $\{A, \varphi\}$ versus $\{C_1, C_2\}$ and $\{B, \delta\}$ versus $\{D_1, D_2\}$

are given. Fig. 5 shows the complete solution of the differential equation and additionally both the homogeneous and the particular solution. At the beginning a very complex motion occurs because time is too short to cause significant damping. After a few oscillations damping lowers significantly the amplitudes of the homogenous part of the motion. Thus more and more the effect of the external driving force dominates the response of the oscillator and it mainly shows the kinesic characteristics of the external source. Especially frequency of the external excitation Ω characterises the oscillation after a sufficiently long time of interaction. The oscillation characteristic of the oscillator – represented by the eigen angular frequency ω – disappears. As it will be discussed in the next chapter the amplitude of the oscillating response of the oscillator to the external source will strongly depend on the capability of the oscillator itself to follow the external drive. There is a maximum amplitude response when the excitation frequency is identical to the resonance frequency of the oscillator.

Tab. 2: Determinants of a driven oscillator

System:

Oscillator			**External source:**	
R [Ω]	C^{-1} [F^{-1}]	L [H]	$U_{0;t}$ [Vs^{-1}]	Ω[s^{-1}]
γ [Nsm^{-1}]	κ [Nm^{-1}]	m [kg]	G_0 [N]	Ω[s^{-1}]
0,5	1	1	2,5	3

Initial conditions

f_0 [A]	$f_{;t0}$ [As^{-1}]
f_0 [m]	$f_{;t0}$ [ms^{-1}]
0,2	1

Parameters and constants of integration

2h	k²	w			
0,5	1	0,96824584			
h²	k²				
0,0625	1				
C_1	C_2		D_1	D_2	
1,08443534	0,2		-0,3018868	0,11320755	
A	φ		B	δ	$\Omega_{Res.}$ [s^{-1}]
1,1027239	0,18237842		0,32241524	0,35877067	0,93541435

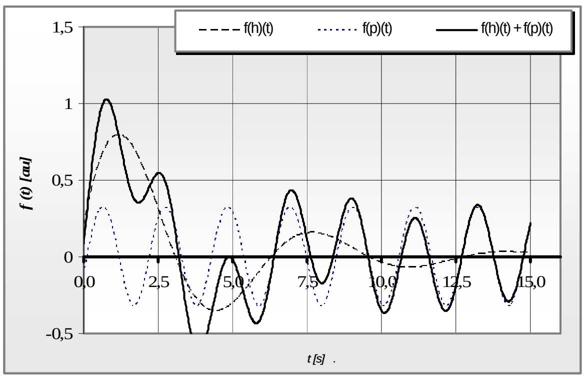

Fig. 5: Oscillation of a driven oscillator

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3.3 Eigen angular frequency, Excitation frequency, Resonance

As discussed in chapter 1 absolutely different physical systems from different acting principles can be described by one type of differential equation: a linear, ordinary differential equation of 2nd order – ODE(O2) – with constant coefficients. From principal it is formulated in equation (1.0-4). In chapter 2 it was shown that this equation describes three different types of motion. Two a-periodic motions (over damped and critically damped oscillators) and a periodic motion (under damped oscillator). This was discussed in detail in chapter 3. It could be shown that oscillation only occurs, when restoring effects dominate friction effects inner the system. Mathematically this is formulated by the constraint of complex eigenvalues of the characteristic polynomial. With this an angular frequency – the so called eigen angular frequency ω – can become defined (2.2-13). It is given once more in equation (4.1). It has to be pointed out that it is strictly to distinguish between this eigen-angular-frequency and the angular frequency Ω of any external excitation with might happen with it so called excitation-angular-frequency. One only will find the eigen-angular-frequency in the mathematical formulation of the homogeneous solution of the describing differential equation. And this homogeneous solution describes the motion of the free, the un-excited system which oscillates with its eigen frequency ν_e (4.1).

$$\omega = +\sqrt{\left| h^2 - k^2 \right|} \quad \rightarrow \quad \nu_e = 2\pi \cdot \omega \tag{4.1}$$

So far external excitation occurs this happens with an excitation-angular-frequency Ω or an excitation frequency $\nu_E = 2\pi.\Omega$. This excitation happens independently from any system parameters of the oscillator. This must be realised at any time! But the reaction of the oscillator to this external excitation strongly depends on Ω_E. As one can see in Fig. 5 the characteristic of the motion of the un-excited oscillator becomes damped away with time and the characteristic of the external excitation more and more determines the motion of the oscillator. But it becomes clear from equations (3.2-1) and (3.2-2) that the amplitude of the external driven part of the oscillation B is dependent from both its amplitude g_0 with the excitation angular frequency Ω and the system parameters of the oscillator h and k. The consequences thereof are formulated in (4.2).

$$B = \frac{g_0}{\sqrt{\left(k^2 - \Omega^2\right)^2 + 4.h^2.\Omega^2}} \tag{4.2}$$

It is clear from the structure of (4.2) that only real roots exist. Thus oscillations with an amplitude *B* are realised for all values of Ω. Keep in mind that the oscillation is only the response of the external excitation with their amplitude of "force" g_0. And g_0 acts as a scaling factor for the oscillation but must not be changed with the amplitude of the oscillation. Structure of equation (4.2) makes clear too that there exists a maximum amplitude dependent from Ω. This maximum amplitude is to derive by calculating the extremum of $B(\Omega)$ by setting the derivation versus Ω equal *0* as shown in (4.3). This defines the so called resonance condition for the excitation angular frequency Ω_{Res} (4.4) and allows calculating the maximum amplitude of the particular solution of the describing differential equation B_{Res}. Resonance frequency ν_{Res} is easy to calculate from $\nu_{Res} = 2\pi.\Omega_{Res}$. In case of small damping (small values for *h*) the resonance amplitude B_{Res} increases dramatically. When damping vanishes completely B_{Res} becomes infinite! It causes violent swaying motion and ends with the catastrophic failure of the oscillator – the so called resonance disaster.

$$\left.\frac{\partial B(h,k,g_0,\Omega)}{\partial\Omega}\right|_{\Omega_{Res}} = \frac{\partial}{\partial\Omega}\left.\left[\frac{g_0}{\sqrt{\left(k^2-\Omega^2\right)^2+4.h^2.\Omega^2}}\right]\right|_{\Omega_{Res}} = 0 \qquad (4.3)$$

$$\Omega_{Res} = \sqrt{k^2 - 2.h^2} \qquad (4.4)$$

$$B_{Res} = \frac{g_0}{2.h.\sqrt{k^2 - 2.h^2}} \qquad (4.5)$$

Another reaction of the oscillator to the external excitation dependent on Ω_E is the phase shift δ between the external excitation and the oscillation mode. A discussed earlier in chapter 3 the negative sign comes from the fact that the oscillation of the physical system is the answer to the excitation. Thus it shows arises after its cause (4.6).

$$\delta \overset{!}{=} -\, atg\ \frac{2.h.\Omega}{k^2-\Omega^2} \qquad (4.6)$$

Long term amplitude of the excited oscillator and phase shift between the excitation and the oscillation response in dependence from Ω/Ω_{Res} are shown in Fig. 6. One can see that excitations with frequencies significantly higher than resonance frequency are filtered effectively. Even at frequencies lower than resonance frequency a filter effect occurs. For excitation frequencies lower than resonance frequency the oscillator follows the excitation with a slight phase shift and slightly lowered amplitude. For excitation frequencies significantly higher than resonance frequency approximately inversely phased oscillation with small amplitudes occurs.

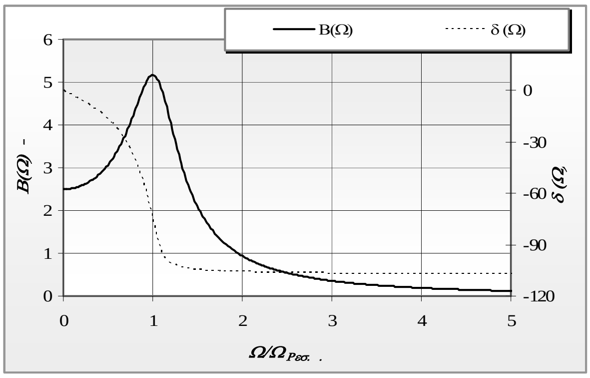

Fig. 6: Amplitude of the particular solution and phase shift between external excitation and oscillation response

Tab. 3: Spezifizierung der Schwingungsgrößen nach dem physikalischen Wirkprinzip

		Mechanical oscillator	Electrical oscillator
Damping	$2h$	$\frac{\gamma}{m}$	$\frac{R}{L}$
Elastic restoring	k^2	$\frac{\kappa}{m}$	$\frac{1}{L.C}$
Eigenvalue	$h^2 - k^2$	$\left(\frac{\gamma}{2.m}\right)^2 - \frac{\kappa}{m}$	$\left(\frac{R}{2.L}\right)^2 - \frac{1}{L.C}$
Eigen angular frequency	ω	$\sqrt{\left\lvert\left(\frac{\gamma}{2.m}\right)^2 - \frac{\kappa}{m}\right\rvert}$	$\sqrt{\left\lvert\left(\frac{R}{2.L}\right)^2 - \frac{1}{L.C}\right\rvert}$
Resonance angular frequency	Ω_{Res}	$\sqrt{\frac{\kappa}{m} - 2.\left(\frac{\gamma}{2.m}\right)^2}$	$\sqrt{\frac{1}{L.C} - 2.\left(\frac{R}{2.L}\right)^2}$
Resonance amplitude	B_{Res}	$\frac{g_0.m}{\gamma.\sqrt{\frac{\kappa}{m} - 2.\left(\frac{\gamma}{2.m}\right)^2}}$	$\frac{g_0.L}{R.\sqrt{\frac{1}{L.C} - 2.\left(\frac{R}{2.L}\right)^2}}$

4 Prospects to wave phenomena

4.1 Oscillation of a string

There is a simple physical system which shows a time dependent but spatial oscillation: a tightened string. It is clear from chapter 2.2.3 and 3.1.3 that specific conditions must be fulfilled to ensure the oscillating behaviour. This is not discussed here. Nevertheless it is easy to see that application of Newton's law of inertia as given in (4.1-1) leads to a balance in forces in *x* and *y* orientation (4.1-2), which are used to derive a new type of describing equation. Notice that *dm* indicates the mass of an infinitesimal small segment of the string. $\Psi\,(x,t)$ indicates its actual dislocation (time dependence) at a specific place along the string (spatial dependence).

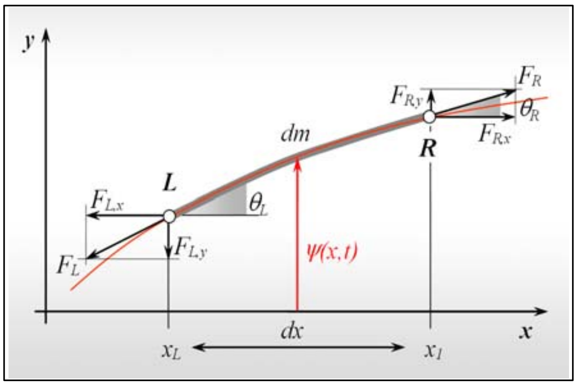

Fig. 7: Segment of an oscillating string

$$\vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n = \sum_{i=1}^{n} \vec{F}_i = \vec{F}_{res} = \vec{F}_{Tr} = m \cdot \vec{a} \qquad (4.1\text{-}1)$$

$$\begin{aligned} -F_L \cdot \cos\theta_L + F_R \cdot \cos\theta_R &= 0 \\ -F_L \cdot \sin\theta_L + F_R \cdot \sin\theta_R &= dm \cdot \frac{\partial\psi^2(x,t)}{\partial t^2} \end{aligned} \qquad (4.1\text{-}2)$$

From this a simple algebraic transformation leads to (4.1-3). Equation (4.1-4) is obtaines when the identity $\tan\alpha = \sin\alpha / \cos\alpha$ is used. In elementary differential calculus it is shown that the tangent of the slope to a curve is identified with the first derivation of the describing function. Thus $\tan\Theta$ can be written as $\partial\Psi(x,t)/\partial x$. With this equation (4.1-5) is formulated.

$$\begin{aligned} F_R &= F_L \cdot \frac{\cos\theta_L}{\cos\theta_R} \\ -F_L \cdot \sin\theta_L + F_L \cdot \frac{\cos\theta_L}{\cos\theta_R} \cdot \sin\theta_R &= dm \cdot \frac{\partial\psi^2(x,t)}{\partial t^2} \end{aligned} \qquad (4.1\text{-}3)$$

$$F_L \cdot \left(-\tan\theta_L + \tan\theta_R\right) = \frac{dm}{\cos\theta_R} \cdot \frac{\partial\psi^2(x,t)}{\partial t^2} \qquad (4.1\text{-}4)$$

$$F_L \cdot \left(-\left.\frac{\partial\psi(x,t)}{\partial x}\right|_{x_L} + \left.\frac{\partial\psi(x,t)}{\partial x}\right|_{x_R} \right) = \frac{dm}{\cos\theta_R} \cdot \frac{\partial\psi^2(x,t)}{\partial t^2} \qquad (4.1\text{-}5)$$

Taylor's serial expansion is used to express $\partial\Psi(x_R,t)/\partial x$ by $\partial\Psi(x_L,t)/\partial x$ as formulated in (4.1-6). To apply this to (4.1-5) leads to (4.1-7).

$$\left.\frac{\partial\psi(x,t)}{\partial x}\right|_{x_R} \cong \left.\frac{\partial\psi(x,t)}{\partial x}\right|_{x_L} + dx \cdot \left.\frac{\partial^2\psi(x,t)}{\partial x^2}\right|_{x_L} \qquad (4.1\text{-}6)$$

$$F_L \cdot dx \cdot \left.\frac{\partial^2\psi(x,t)}{\partial x^2}\right|_{x_L} = \frac{dm}{\cos\theta_R} \cdot \frac{\partial^2\psi(x,t)}{\partial t^2} \qquad (4.1\text{-}7)$$

With the identity $dm = \rho \cdot dV$ and $dV = A \cdot dx$ one obtains: $dm = \rho \cdot A \cdot dx$. One can assume that the norm of F_L and F_R are similar to each other and to the tension force of the string in its rest position F_0. Notice that F_0/A defines the tensile strength σ inner the string. Division by ρ leads to σ/ρ on the left hand side of (4.1-7). Its Dimension is 1(m/s)² - the square of the dimension of a velocity. Physical this describes the velocity of the actual displacement inner the string[d]. Commonly it is indicated with c^2 Therefore one writes $\sigma/\rho = c^2$. assumption of small displacements leads to small numbers for any angel θ. Thus one can approximate cos $\theta \cong 1$. From all this the wave equation of an oscillationb string can be written ((4.1-8). This is a new type of differential equation dependent from minimum two variables and from 2nd order in both time *t* and space *x*. A new technique to solve this *partial differential equation* has to be used. It is not part of this paper to explain this in detail. But in any case when the differential operator can be written as a sum of individual differentials in one variable only, separation helps to find the solution. Here one can write the differential operator as: $[c^2.\partial^2/\partial x^2 - \partial^2\Psi/\partial t^2].\Psi(x,t) = 0$. Thus a separation $\Psi(x,t) = X(x).T(t)$ works. Equation (4.1-8) modifies to $c^2.T(t).d^2X(x)/dx^2 = d^2T(t)/dt^2] = K^2$. K^2 is the constant of separation. After division by $X(x).T(t)$ one obtains two ordinary differential equations (4.1-9).

$$c^2 \cdot \left.\frac{\partial^2\psi(x,t)}{\partial x^2}\right|_{x_L} = \frac{\partial^2\psi(x,t)}{\partial t^2} \tag{4.1-8}$$

$$\frac{d^2X(x)}{\partial x^2} = \frac{K^2}{c^2}\cdot X(x); \qquad \frac{d^2T(t)}{\partial t^2} = K^2\cdot T(t) \tag{4.1-9}$$

Thus as a consequence of separation two linear ordinary homogeneous differential equations with constant coefficients for *X(x)* and *T(t)* are generated. The corresponding eigenvalues are $\xi_{1;2} = \pm (K^2/c^2)^{1/2}$ and $\delta_{1;2} = \pm K$. Oscillation only occurs when $K^2 < 0$ and the eigenvalues become complex therefore. With $k^2 = |K^2|$ equation (4.1-10) is the general solution of the wave equation of an oscillating string.

$$\psi(x,t) = [A\cdot\sin(kt) + B\cdot\cos(kt)] + [C\cdot\sin(k/_c\, x) + D\cdot\cos(k/_c\, x)] \tag{4.1-10}$$

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[d] In case of a tightened string it is not easy to give a descriptive interpretation of *c*. But the constraint of a fixation at both sides of the string was not used anywhere during the deduction of the characterising equation. This indicates that it is not from any importance for the description of the phenomenon “wave”. In case of a fixationat on e side only a simpel interpretation of *c* is possible. The string can be moved at the free end. And it is obvious that a wave like movement of the string occurs when an arbitrary stimulation happens. Wave crests move with a velocity c along the string. In case of a longitudinal wave propagation as it happens in a tube of a wind instrument *c* is equalt to the acoustic velocity. For electromagnetic waves *c* is the light velocity.

Specifics of fixation become importaint when a concrete system is investigated. They define the boundaries of the phenomenon and enforce the describing system of functions. Therefore the conditions of fixation become essential when the wave equation is to be solved.

The constants of integration are specified by the definition of the fixation at the right and at the left side (two conditions) and two initial conditions defined by the displacement function $f(x) = \Psi(x,0)$ and an initial status of movement $g(x) = \Psi_{,t}(x,0)$, with $\Psi_{,t}(x,0)$ indicating first derivation versus time of the solving Ψ – function. From Fourier's theory one obtains an infinite series of trigonometric functions (4.1-11).

$$\psi(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{L}x \cdot \left[A_n \sin\frac{n\cdot\pi\cdot c}{L}t + B_n \cos\frac{n\cdot\pi\cdot c}{L}t\right]$$

$$A_n = \frac{2}{n\cdot\pi\cdot c}\cdot\int_0^L g(x)\sin\frac{n\pi}{L}x\cdot dx \qquad (4.1\text{-}11)$$

$$B_n = \frac{2}{L}\cdot\int_0^L f(x)\sin\frac{n\pi}{L}x\cdot dx$$

4.2 More dimensional wave phenomena

An oscillation string represents a one dimensional wave phenomenon. It is easy to broaden the concept to two and three dimensional phenomena as the oscillation of a membrane or an arbitrary three dimensional object. So far a Cartesian description is chosen the one dimensional spatial derivation $\partial^2/\partial x^2$ must be supplemented with $\partial^2/\partial y^2$ and $\partial^2/\partial z^2$ respectively. In general a non Cartesian description will be necessary, e.g. when cylindrical or spherical problems have to be described. Notice that lots of different coordinate systems are chosen to describe problems in a geometry defined by their own coordinate surfaces (e.g. toroidal, parabolic, hyperbolic surfaces – and others). In these geometries the second spatial derivation operation as a scalar differential operator has to be formulated specific to the relevant geometry. This is a sophisticated mathematical procedure and is not described here. But it is pointed out that the generalised differential operator is called *Nabla – Operator*. It is indicated as Δ. And it is aim of this chapter that the reader is able to read a more dimensional wave equation and to understand the symbol used. Thus the wave equation formulated for a more dimensional problem is formulated in (4.2-1).

$$c^2\cdot\Delta\psi(\vec{x},t) = \frac{\partial^2\psi(\vec{x},t)}{\partial t^2} \qquad (4.2\text{-}1)$$

The Nabla formulation is used in lots of differential equations in physics whe a second spatial derivation is needed. For example the wave equations of the Maxwell theory (4.2-2), Schrödinger's equation (4.2-3) and thermal conductivity equation (4.2-4) are given. It is noticed that thermal conductivity equation is not a wave equation but a so called transport equation.

$$c^2 \cdot \Delta\vec{E}(\vec{x},t) = \frac{\partial^2 \vec{E}(\vec{x},t)}{\partial t^2}$$
$$c^2 \cdot \Delta\vec{B}(\vec{x},t) = \frac{\partial^2 \vec{B}(\vec{x},t)}{\partial t^2} \qquad (4.2\text{-}2)$$

$$\left(-\frac{\hbar^2}{2.m}.\Delta + V(\vec{x})\right).\psi(\vec{x},t) = i.\hbar.\frac{\partial\psi(\vec{x},t)}{\partial t} \qquad (4.2\text{-}3)$$

$$\frac{\partial T(\vec{x},t)}{\partial t} = \frac{\lambda(T)}{\rho(T)\cdot c_p(T)} \cdot \Delta T(\vec{x},t) = a(T)\cdot\Delta T(\vec{x},t) \qquad (4.2\text{-}4)$$

All these equations show a second derivation in minimum one variable. From principle this leads to a solution for the separated function in this variable which shows an oscillation.

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5 Harmonic oscillations - some more literature

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6 References

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