W. Hohenauer:

Physical modelling based on first order differential equations (first order ODE´s)

Abstract

An introduction in techniques to solve first order ODE´s is given. Barometric formula, charging and discharging of a capacitor including its application for AMV, magnetic induction and growth effects are discussed. Physics of magnetic induction motivates to extend modelling capabilities in oscillation phenomena.

1 Introduction

Lots of every day occurrences are driven from a simple active principle: “the change of the effect is proportional to the actual magnitude of the effect”. Some examples therefore are:

> The reaction rate of a chemical reaction (= change of concentration of an educt) often depends on the concentration of the evolved educts.
>
> The number of radioactive disintegration processes of an instable isotope strongly depends on the number of particles of the isotope.
>
> The reproduction of a population depends on the number of its individuals.

The mathematical formulation of phenomena like this leads to ordinary differential equations (ODE) of first order. In case of linear equations their solutions can be given – even when inhomogeneities occur (c.f. Capt. 2.1). But the analytical formulation depends on the inhomogeneity itself and their antiderivative must be found. Nevertheless: the knowledge of the structure of the solution of the descriptive ODE allows its numerical examination in any case.

In this text an overview is given how to solve linear, non-homogneous first order ODE´s with a variable coefficient. Some examples from physics are given: barometric formula, charge and discharge of a capacitor, the current in a magnetic coil when switching it on or off, and a few case studies to problems of natural growth.

2 First order linear, non-homogenous differential equations with variable coefficient

2.1 Mathematical model

The active principle: “The change of an observable is proportional to its actual magnitude” can easily be formulated by a mathematical equation. Most effects following this principle are dependent either from time or from a space variable. To avoid any specification at this moment, a generalized variable *q* is used to describe the dependency of the observable. The observable itself is identified with the describing function *f(q)*. The change of the observable is described be its first derivation *df(q)/dq*. The proportionality between these two quantities is defined by a factor *a(q)*, which mostly is a function from the variable *q* too (variable coefficient). Additionally the observed phenomenon can be inflected by effects which are from principle independent from the acting principle. Influences like this are incorporated by a source term *g(q)*. Mathematically *g(q)* defines the inhomogeneity of the problem, and it has to be noticed once more that *f(q)* and *g(q)* are absolutely independent from each other! With these quantities the acting system is described by equation (2.1-1).

$$\frac{df(q)}{dq} = a(q) \cdot f(q) + g(q) \tag{2.1-1}$$

Formally from (2.1-1) a differential operator [*d/dq –a(q)*] can be derived. To apply this operator to the describing function *f(q)* leads to an equivalent representation of the describing equation as given in (2.1-2).

$$\left[\frac{d}{dq} - a(q)\right] f(q) = g(q) \tag{2.1-2}$$

The differential operator [*d/dq –a(q)*] determines the structure of the solution of the mathematical problem. Linear algebra provides the principal independence of this structure from the inhomogeneity *g(q)*. This consequences the mathematical method to find the solving function *f(q)* of equations (2.1-1). Firstly, neglecting the inhomogeneity – the solution of the homogeneous differential equation $f^{(h}(q)^{)}$ is calculated. In a second step the inhomogeneity is used to construct a particular solution $f^{(p)}(q)$ specific for the structure of the source term *g(q)*. Mathematics shows that the complete solution of the differential equation is the sum of both, the homogeneous and the particular solution.

From principal the way to find the solution of a first order differential equation is to integrate the equation. Therefore an constant of integration *C* becomes part of the solving function *f(q)*. It becomes determined when an initial condition (time dependent problems) or a boundary condition (spatial problem) is defined. This specifies the general problem to a specific application. The mathematical formulation of this condition gives an additional equation and leads to the examination of the constant of integration *C*.

2.2 Solution of the homogeneous differential equation

Ignoring the source term *g(q)*, (formally: to set $g(q) = 0$), specifies to the homogeneous differential equation and its homogeneous solution $f^{(h)}(q)$ which exclusively captures the structure of the homogenous differential operator. Consequently this leads to zero at the right side of equation (2.1-2) – as shown in (2.2-1). It is part of a later step in finding the complete solution of the problem to construct a particular solution $f^{(p)}(q)$ which considers the deviation from this zero-quantity. Simple algebraic transformation of (2.2-1) leads to (2.2-2). The notation $f^{(h)}(q)$ indicates that only the homogeneous equation is fulfilled.

$$\left[\frac{d}{dq} - a(q)\right] f^{(h)}(q) = 0 \qquad (2.2\text{-}1)$$

$$\frac{df^{(h)}(q)}{f^{(h)}(q)} = a(q)\cdot dq \qquad (2.2\text{-}2)$$

Integration of (2.2-2) leads to $ln|f^{(h)}(q)|$ on the left side. On the right side this leads to an integral $\int a(q).dq + \hat{C}$. Thus equation (2.2-3) is only defined for positive definite functions $f^{(h)}(q)$. To apply the exponential function and using the definition $exp(\hat{C}) := C$ leads to equation (2.2-3). Consider that this is compatible with the restriction to positive definite solution space because $|f^{(h)}(q)| = f^{(h)}(q)\ \forall\ exp[\int a(q).dq + \hat{C}]$.

$$f^{(h)}(q) = C\cdot e^{\int a(q)\cdot dq} \qquad (2.2\text{-}3)$$

As mentioned earlier $f^{(h)}(q)$ only captures the principal structure of the acting principle. So far no inhomogeneity is necessary to formulate the observed effect (to calculate barometric formula can be drawn back to a problem like this), the principal solution of the problem $[d/dq - a(q)].f^{(h)}(q)=0$ is found. Only specification of the constant of integration C has to be done. In any other case the source term $g(q)$ has to be used to construct a particular solution $f^{(p)}(q)$. The idea is, to find one specific function $f^{(p)}(q)$ fulfilling that $[d/dq - a(q)].f^{(p)}(q)=g(q)$. Mathematics shows that the sum of $f^{(h)}(q)+f^{(p)}(q)$ represents the complete solution $f(q)$ of the problem as specified by the equation $[d/dq - a(q)].f(q)=g(q)$. Once more: specification of the constant of integration C will to be done by fulfilling an initial or boundary condition. This is described in the following chapters.

2.3 Solution of the non-homogenous differential equation

To construct a partial solution $f^{(p)}(q)$ specific for the source term $g(q)$ both is used: first the homogeneous solution $f^{(h)}(q)$ considering the principal mathematical structure of the problem and the specific inhomogeneity $g(q)$ describing the additional external influences to the problem, which are not captured by the acting principle itself. The principal mathematical structure of the problem is considered by a generalisation of the solution of the homogeneous equation (2.2-3) in that way, that the constant of integration of the homogeneous equation itself is generalised to be a function of q in the non-homogenous problem: $C \to C(q)$. This is the only way which enables to apply the structure of the homogeneous solution and to provide an additional possibility to vary this solution to a more general one – which considers the specifics of the source term $g(q)$. Thus the method is called “variation of the constant”. Equation (2.2-3) changes to equation (2.3-1). From this (2.3-2) can be calculated. With this and with the application of the differential operator (2.1-2) at (2.3-1) gives the differential equation for $C(q)$ (2.3-3).

$$f(q) = C(q) \cdot e^{\int a(q) \cdot dq} \qquad (2.3\text{-}1)$$

$$\frac{df(q)}{dq} = \frac{dC(q)}{dq} \cdot e^{\int a(q) \cdot dq} + C(q) \cdot a(q) \cdot e^{\int a(q) \cdot dq} \qquad (2.3\text{-}2)$$

$$\frac{dC(q)}{dq} \cdot e^{\int a(q) \cdot dq} = g(q) \qquad (2.3\text{-}3)$$

To multiply (2.3-3) with *exp*[-∫*a(q).dq*].*dq* gives equation (2.3-4) for *dC* and after integration equation (2.3-5) with the constant of integration *K*. With the function *C(q)* the complete solution of the non-homogenous problem is fount.

$$dC(q) = \left[g(q) \cdot e^{-\int a(q) \cdot dq} \right] dq \qquad (2.3\text{-}4)$$

$$C(q) = \int \left[g(q) \cdot e^{-\int a(q) \cdot dq} \right] dq + K \qquad (2.3\text{-}5)$$

$$f(q) = K \cdot e^{\int a(q) \cdot dq} + \left[\int \left(g(q) \cdot e^{-\int a(q) \cdot dq} \right) dq \right] \cdot e^{\int a(q) \cdot dq} \qquad (2.3\text{-}6)$$

Equation (2.3-6) summarizes the homogeneous solution $f^{(h)}(q)$ = *K.exp*[∫*a(q).dq*].*dq* with a re-named constant of integration *K* and the partial solution $f^{(p)}(q)$ = {∫*g(q).exp*[-∫*a(q).dq*].*dq}.exp*[∫*a(q).dq*]. The integral function ∫*g(q).exp*[-∫*a(q).dq*].*dq* is a generalized constant of integration specific for the inhomogeneity *g(q)*. The integral expression for $f^{(p)}(q)$ cannot be calculated analytically in any case. So far no antiderivative exists numerical methods must be applied. To specify the constant of integration *K* an initial or boundary condition must be fulfilled. To do this, for an arbitrary argument q_i the value of the function $f(q_i)$ must be known. Mostly these values are given for $q_i = q_0 = 0$. In Fig. 1 and Fig. 2 examples for a homogeneous problem and a non-homogenous problem with a periodic disturbance are given. From the homogeneous problem, which describes a system of spatial dependence, one learns that a strictly positive proportionality causes an exponential increase of the solving function. This makes clear that any behaviour like this will consequence the collapse of the system. Strictly negative proportionalities cause a decrease of the describing function to zero. The example shown in Fig. 2, which describes a system dependent from time, makes clear that a negative proportionality in combination with a non vanishing inhomogeneity can cause a non-zero and finite describing function of the acting system. This is to interpret as a “sustainable” behaviour of the system which guarantees a “surviving” of the system for a long period of time.

W. Hohenauer:

f ′ = a . f(t); a = const.

$K = y_0$	1,00	1,00
a	0,10	-0,10

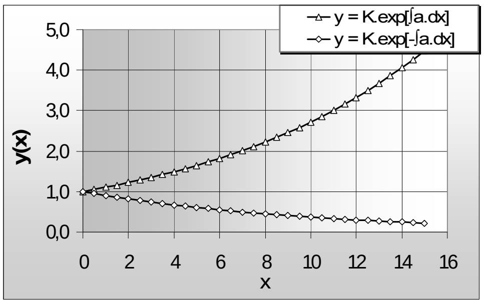


Fig. 1: Solution of homogeneous problem referring to data given in the table.

f ′ = a.f(t) + g₀.sin(wt.); a = const.

a	-0,50		
π /Hz	0,25	π	1,57
g_0	8,00		
$K; f_0 = 0$	2,31		

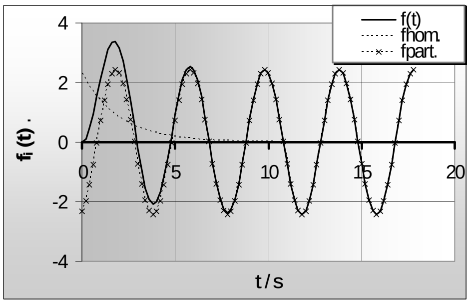


Fig. 2: Solution of non-homogenous problem referring to data given in the table.

3 Examples

3.1 Barometric formula

Pressure p is defined as the quotient of a force F and a surface A as given in equation (3.1-1).

$$p = \frac{F}{A} \tag{3.1-1}$$

The relevant force in the current problem is the weight F_g of a layer of gas above surface in a specific height y as shown in Fig. 4. Weight itself is the product from mass m and acceleration due to gravity g. Mass can be calculated from the product of a considered volume V and the density ρ of its content. It is useful to notice that the density of a gas changes with the height. The reason therefore is the compressibility of any gas – in opposite to liquids and solids. Therefore it will become important to calculate the dependency of the density $\rho(y)$ of a layer of gas from its height. Thermodynamics teaches a strict interdependence of the thermodynamic state variables: pressure p, volume V, und temperature T . For a simple estimation the state equation of a ideal gas (3.1-2) can be used.

$$p \cdot V_m = R \cdot T \tag{3.1-2}$$

It ignores both the volume of the gas molecules themselves and any adhesive interaction between these molecules. The physical quantities used are:

Gas constant R
R = 8,314 J/mol.K

Ambient pressure p_0
$p_0 = 1{,}013\ bar$

Volume of 1 mol of any gas at ambient conditions V_m
$V_m = 22{,}4\ ltr$ or $22{,}4 \ . \ 10^{-3}\ m^3$

The situation is illustrated in Fig. 3. Notice that the definition of dp is: $dp = p(y+dy) – p(y)$. Because $p(y)$ is larger than $p(y+dy)$ this difference becomes negative! This is the mathematical consequence of the fact that p(y) decreases with increasing height. Using (3.1-1) and the facts that $F_g = dm.g;$ $dm = \rho.dV;$ and $dV = A.dy$ the pressure dp caused from the infinitesimal thin layer between the heights y and $y+dy$ is calculated by (3.1-3)

$$-dp = \frac{F_g}{A} = \rho(y)\cdot g\cdot dy \qquad (3.1\text{-}3)$$

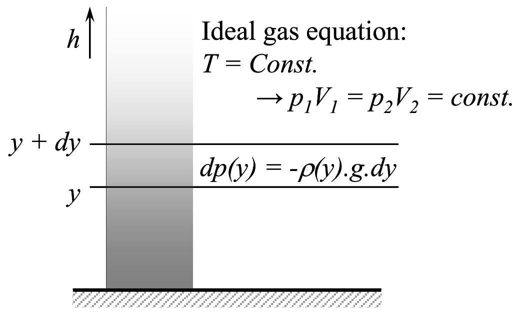


Fig. 3: Decrease of pressure in dependence of height.

Fig. 3 demonstrates the need to find a mathematical expression for the density $\rho(y)$ of a layer of gas. To derive a simple model from the ideal gas equation (3.1-2) a constant temperature all over the layer of gas is assumed (isothermal conditions: T = *const.*) With this assumption equation (3.1-2) simplifies to Boyle-Mariotte´s law (3.1-4).

$$p_1\cdot V_1 = p_2\cdot V_2 = const. \qquad (3.1\text{-}4)$$

The definition of the density $\rho := m/V$ and the use of ambient conditions p_0 and ρ_0 as a reference status leads to equation (3.1-5). It calculates density in an arbitrary height $\rho(y)$ as a function of the pressure $p(y)$ in this height (3.1-6).

$$\frac{V_1}{V_2} = \frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} \qquad (3.1\text{-}5)$$

$$\frac{p(y)}{p_0} = \frac{\rho(y)}{\rho_0} \rightarrow \rho(y) = \frac{\rho_0}{p_0}\cdot p(y) \qquad (3.1\text{-}6)$$

With (3.1-6) and Fig. 4 equation (3.1-7) can be formulated. It is a linear, homogeneous first order differential equation with a constant coefficient with a solution as given in (2.2-3) or in (2.3-6) with a vanishing inhomogeneity. One has to identify as follows:

- $q \rightarrow y$
- $f(q) \rightarrow p(y)$
- $a(y) = (-\rho_0.g)/p_0$
- $g(y) = 0$

Equations (2.2-3) or (2.3-6) specify to (3.1-8). The boundary condition $p(y=0) = p_0$ specifies $K = p_0$. The density of air under ambient conditions is $\rho_0 = 1{,}29\ \mathrm{kg/m^3}$.

$$-dp(y) = \frac{\rho_0 \cdot g}{p_0} \cdot p(y) \cdot dy \qquad (3.1\text{-}7)$$

$$p(h) = p_0 \cdot e^{-\frac{\rho_0 \cdot g}{p_0} \cdot h} \qquad (3.1\text{-}8)$$

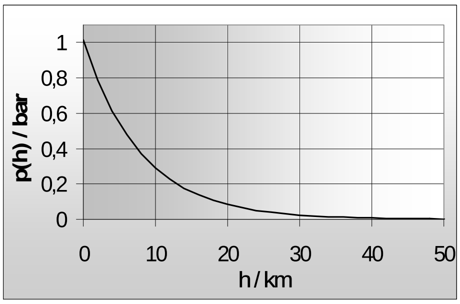


Fig. 4: Decrease of pressure in dependence from height under assumption of isothermal conditions.

Isothermal assumptions and ideal gas conditions lead to a linear first order ODE. As a result the curve *p(h)* as shown in Fig. 4 has the typical exponential decreasing shape. More realistic considerations as they are necessary in metrology have to implement full thermodynamics and realistic temperature profiles of the atmosphere in their models – which makes the problem much more complicated.

3.2 Charging and discharging of a capacitor under DC and AC conditions

The electric charge stored in a capacitor is proportional to the voltage measured at its connections. The proportional factor is called capacitance *C*. The interrelation is given in (3.2-1). The electric charge is the time-integral of the electric current flowing in the circuit. So far the capacitor is completely uncharged at a time t_0 the integral in equation (3.2-2) which calculates the change of charge $\Delta Q = Q(t_x) - Q(t_0)$ within an observation time $\Delta t = t_x - t_0$ describes the total charge stored in the capacitor after a time t_x. Combination of these two equations gives (3.2-3) and after a derivation $^{d}/_{dt}$ equation (3.2-4)

$$Q(t) = C \,.\, U_C(t) \qquad (3.2\text{-}1)$$

$$\Delta Q\Big|_{t_0}^{t_x} = \int_{t_0}^{t_x} I(t)\cdot dt \qquad (3.2\text{-}2)$$

$$U_C(t) = \frac{1}{C}\cdot\int_{t_0}^{t_x} I(t)\cdot dt \qquad (3.2\text{-}3)$$

$$I(t) = C\cdot\frac{dU_C(t)}{dt} \qquad (3.2\text{-}4)$$

Any electric circuit has a non vanishing electric resistivity *R*. So formally an electric connection scheme as shown in Fig. 5 describes the acting system. Ohm´s law (3.2-5) is used to calculate the voltage drop over the resistivity.

$$U(t) = I(t) \,.\, R \qquad (3.2\text{-}5)$$

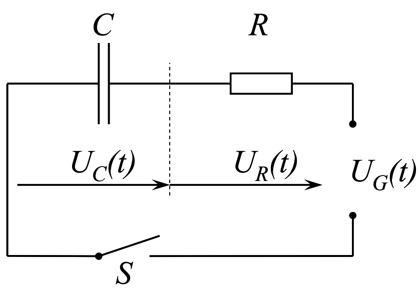


Fig. 5: Serial connection of an electric resistivity and a capacitor

The basic law to formulate the model describing the transient voltage of the capacitor as shown in the connection scheme of Fig. 6 is Kirchhoff's law (3.2-6): the sum of all voltage drops in a mesh U_i is equal to the external (generating) voltage U_G.

$$\sum_{i=1}^{n} U_i(t) \;=\; U_G(t) \tag{3.2-6}$$

This means that the voltage drops over the capacitor U_C and the resistivity U_R compensate the generating voltage U_G. Whenever a current charges or discharges a capacitor its charge changes. From (3.2-1) this causes a time dependent voltage over the capacitor during charging or discharging processes. Therefore U_C has to be set as a time-dependent function – even when the driving external voltage is a DC voltage. This makes clear that U_R is to be assumed as a time-dependent function too. This is considered in the balancing equation – derived from Kirchhoff´s law – given in (3.2-7)

$$U_C(t) + U_R(t) \;=\; U_G(t) \tag{3.2-7}$$

Using Ohm´s law (3.2-5) and the expression (3.2-4) to express the current I(t) in the circuit one can specify equation (3.2-7) as (3.2-8)

$$U_C(t) + RC \cdot \frac{dU_C(t)}{dt} \;=\; U_G(t) \tag{3.2-8}$$

Algebraic transformation in the standard representation as given in (2.1-1) leads to (3.2-9).

$$\frac{dU_C(t)}{dt} \;=\; -\frac{1}{RC}U_C(t) + \frac{1}{RC}U_G(t) \tag{3.2-9}$$

With the identifications: q → t, f(q) → U_C(t), a(q) →a(t) = -1/R.C and g(t) = U_G(t) one obtains (3.2-10)

$$U_C(t) = K \cdot e^{-\int \frac{dt}{RC}} + \left[\int \left(\frac{U_G(t)}{RC} \cdot e^{\int \frac{dt}{RC}} \right) dt \right] \cdot e^{-\int \frac{dt}{RC}} \tag{3.2-10}$$

3.2.1 DC power supply

Under DC charging conditions $U_G(t)$ is a constant e.g. representing the voltage of a battery U_B. Equation (3.2-10) simplifies to (3.2-11) and leads to equation (3.2-12).

$$U_C(t) = K \cdot e^{-\int \frac{dt}{RC}} + \frac{U_B}{RC} \cdot \left[\int \left(e^{\int \frac{dt}{RC}} \right) dt \right] \cdot e^{-\int \frac{dt}{RC}} \qquad (3.2\text{-}11)$$

$$U_C(t) = K \cdot e^{-\frac{t}{RC}} + U_B \qquad (3.2\text{-}12)$$

Assuming a completely discharged capacitor at the time $t_0 = 0$ the initial condition $U_C(t = 0) = 0$ is valid. One obtains $K = -U_B$ resulting equation (3.2-13)

$$U_C(t) = U_B \cdot \left[1 - e^{-\frac{t}{RC}} \right] \qquad (3.2\text{-}13)$$

3.2.2 Discharging of a capacitor

To discharge a capacitor the power supply is substituted by a closed switch enabling a discharging current. This construction is described mathematically by setting $U_B = 0$. With this equation (3.2-12) can be used to calculate the solution. As initial condition one formulates $U_C(t = 0) = U_{C,0}$. The solution of the problem is (3.2-14)

$$U_C(t) = U_{C;0} \cdot e^{-\frac{t}{RC}} \qquad (3.2\text{-}14)$$

f ' = a.f(t) ; a = const.

a:=-1/RC	*-2,00000*
R	*5,00*
C	*0,10000*
U_B	*12,00*

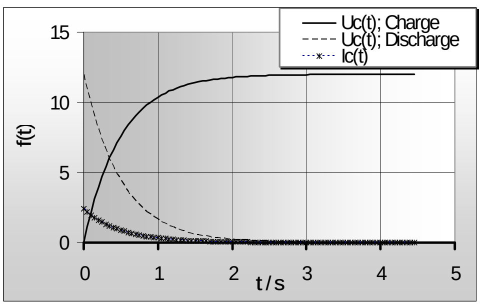


Fig. 6: Charging and discharging behaviour of a capacitor under DC conditions.

From Ohm´s law (3.2-5) and equations (3.2-13) and (3.2-15) one obtains the time dependent current $|I_C(t)|$ for the charging and discharging processes (3.2-15). The results are shown in Fig. 6 too.

$$\left|I_C(t)\right| = \frac{U_C}{R}\cdot e^{-\frac{t}{RC}} \qquad (3.2\text{-}15)$$

3.2.3 Application: A-stable Multivibrator - AMV

From physical reasons the description of the commutation of an a-stabile multivibrator (AMV) as shown in Fig. 7 needs to set $U_C(t_{i,0} = 0) = -U_B$ for all commutations after the first one. Then (3.2-12) gives $K = -2U_B$ and the solution of (3.2-16) results. A transistor typically changes from an open switch in a closed one when its Basis – Emitter voltage U_{BE} reaches from zero to ~0,6V. To estimate the commutation frequency of an AMV the capacitor connected with the full-off (=locked) transistor approximately is assumed to reach a voltage $U_C = U_{BE} = 0$V. The time required to achieve this voltage is t_S. To specify (3.2-16) to this leads to a circuit time $t_S = R.C.ln2$ and to a frequency of the AMV $\nu = 2/[R.C.ln2]$.

$$U_C(t) = U_B\cdot\left[1-2\cdot e^{-\frac{t}{RC}}\right] \qquad (3.2\text{-}16)$$

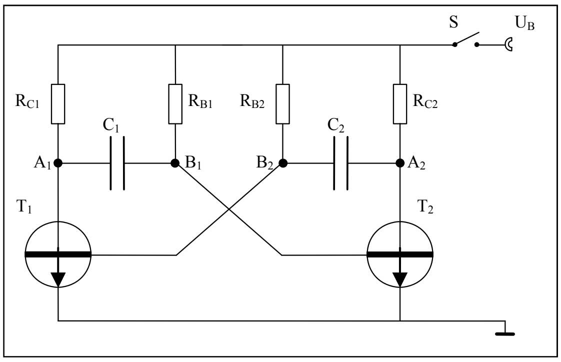


Fig. 7: Connection scheme of an astabile multivibrator – AMV

3.2.4 Sinusoidal supply voltage of a capacitor

$U_G(t)$ is assumed to be a sinusoidal: $U_G(t) = U_0 \,.\, \sin(wt)$. Equation (3.2-10) modifies to (3.2-13).

$$U_C(t) = K \cdot e^{-\int \frac{dt}{RC}} + \frac{U_0}{RC}\left[\int\left(\sin(\omega t)\cdot e^{\int \frac{dt}{RC}}\right)dt\right]\cdot e^{-\int \frac{dt}{RC}} \qquad (3.2\text{-}18)$$

The solution of the integral can be found in mathematical handbooks – for example in Bronstein’s Taschenbuch der Mathematik. With some easy algebraic transformations one will find (3.2-19).

$$\int\left(\sin(\omega t)\cdot e^{\int \frac{dt}{RC}}\right)dt = RC\cdot e^{\frac{t}{RC}}\cdot\frac{[\sin(\omega t) - \omega RC\cdot\cos(\omega t)]}{1+(\varpi RC)^2} \qquad (3.2\text{-}19)$$

From (3.2-18) and (3.2-19) equation (3.2-20) results. With the initial condition $U_C(t = 0) = 0$ one obtains K as given in equation (3.2-21). The result is shown in Fig. 8.

$$U_C(t) = K\cdot e^{-\frac{t}{RC}} + U_0\cdot\frac{\sin(\omega t) - \omega RC\cdot\cos(\omega t)}{1+(\omega RC)^2} \qquad (3.2\text{-}20)$$

$$K \;=\; \frac{U_0\cdot\omega RC}{1+(\omega RC)^2} \qquad (3.2\text{-}21)$$

$f' = a.f(t) + g_0.\sin(\omega\, t.)$; a = const.			
a:=-1/RC			
R	1000		
C	0,001	a = -1/RC	-1,00000
ω	1	ω	6,28
$g_0 = U_0$	10,00		
K	1,55223		

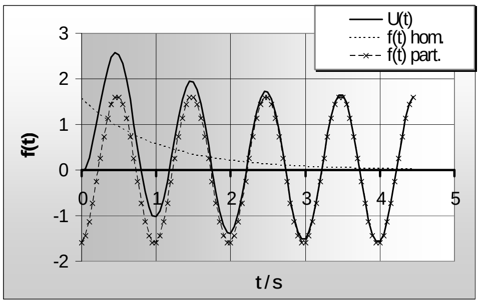


Fig. 8: R-C serial driven by a sinusoidal voltage.

3.3 Transient current of a magnetic coil under DC and AC conditions

Phenomenology of magnetism shows that any change of the magnetic field seen by a coil causes the induction of a voltage. The source of the magnetic field may be a permanent magnet or another coil. To get a simple idea from the phenomenon consider a permanent magnet which is moved through a coil with different velocities. In Fig. 9 the induced voltage from three experiments is shown. A detailed investigation shows that the area between the measured voltage curves and the axis is a constant quantity which only depends from the:

magnetizing field of the permanent magnet H

cross section of the coil A

number of turns n

with respect to the physical units – a number μ_0, called magnetic permeability.

The area between the measured voltage curves and the axis can be calculated by the time integral of the induced voltage $\int U_{ind.}(t).dt$ - as shown in the figure too. This areas is called "voltage surge"

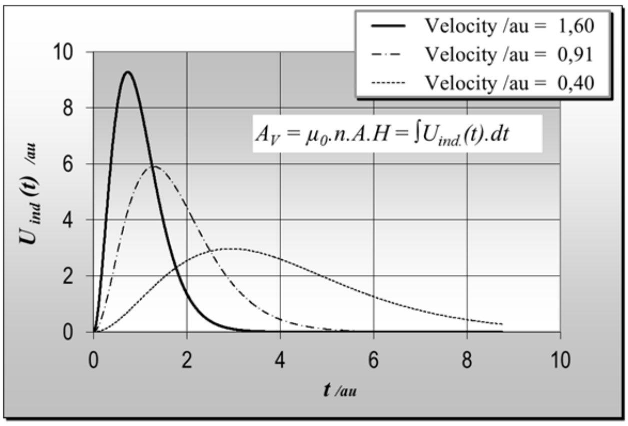


Fig. 9: Induced voltage and voltage surge; quantities in au (arbitrary units)

So far the magnetic magnetizing field H comes from another coil which is flown from an electric current I, H

directly depends from the current itself and a geometry function f_G summarizing geometric details as number of turns, length and cross section, and so on. With this and the introduction of the magnetic quantities: magnetic field B and magnetic flux Φ the set of describing equations (3.3-1) can be given. Additionally another formulation for the voltage surge can be given, which uses the current in a magnetic coil to describe the induction effect (3.3-2). From principle it does not matter if an “external” coil causes a magnetic induction or if the electric current in one specific coil causes their magnetic field. The only difference is that the induced voltage caused from an external source drives an electric current in the other coil responsible for a magnetic response which weakens the effect of the external source. This is considered by the sign of the voltage surge.

$$\begin{aligned} H(t) &= f_G \cdot I(t) \\ B(t) &= \mu_0 \cdot H(t) \\ \Phi(t) &= B(t) \cdot A \end{aligned} \qquad (3.3\text{-}1)$$

$$A_V = \mu_0 \cdot A \cdot n \cdot \left[f_G \cdot \int \frac{dI(t)}{dt} \cdot dt \right] = \int U_{ind.}(t) \cdot dt \qquad (3.3\text{-}2)$$

The product $\mu_0.A.n.f_G$ defines the so called “magnetic inductivity” L of the coil (3.3-3). Using these three equations both, the magnetic induction law (3.3-4) and the magnetic voltage drop of a coil (3.3-5) can be given. The negative sign in (3.3-4) considers the weakening of the external source as mentioned earlier.

$$L = \mu_0 \cdot A \cdot n \cdot f_G \qquad (3.3\text{-}3)$$

$$U_{ind.}(t) = -n \cdot \frac{d\Phi(t)}{dt} \qquad (3.3\text{-}4)$$

$$U_L(t) = U_{ind.}(t) = A_V = L \cdot \frac{dI(t)}{dt} \qquad (3.3\text{-}5)$$

Any coil can be seen as an ideal inductivity L in series with an electric resistor R. So formally the connection scheme as shown in Fig. 10 shows the acting system. Kirchhoff’s law (3.2-6) is used to balance the voltage drops of the inductivity and the resistor with the external (generating) voltage U_G. This means that the voltage drops over the inductivity L: U_L and the resistivity R: U_R compensate the generating voltage U_G. Whenever a current flows through the coil a magnetic response happens. From equation (3.3-5) this causes a time dependent voltage over the coil. Therefore

U_L has to be set as a time-dependent function – even when the driving external voltage is a DC voltage. This makes clear that U_R is to be assumed as a time-dependent function too. This is considered in the balancing equation – derived from Kirchhoff´s law – given in (3.3-6) and (3.3-7)

3.3.1 To switch on a magnetic coil

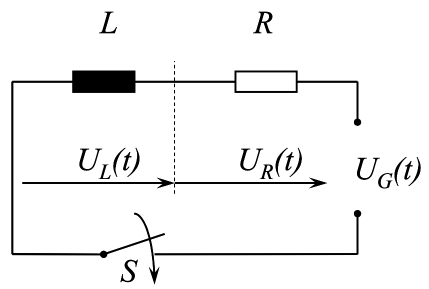


Fig. 10: Connection scheme: to switch on a magnetic coil with a serial resistor

With $U_L = L.dI/dt$ and Kirchhoff's law (3.2-1) the differential equation for the time dependent current can be formulated as equation (3.3-1)

$$L \cdot \frac{dI(t)}{dt} + R \cdot I(t) = U_G(t) \qquad (3.3\text{-}6)$$

$$\frac{dI(t)}{dt} = -\frac{R}{L} \cdot I(t) + \frac{U_G(t)}{L} \qquad (3.3\text{-}7)$$

With the identifications: $q \rightarrow t$, $f(q) \rightarrow I(t)$, $a(q) \rightarrow a(t) = -R/L$, $g(t) = U_G(t)/L$, and $U_G(t) = const.$ equation (2.3-6) results in (3.3-8). Because there is no current before the switch is closed at $t = 0$ as initial condition $I(t=0) = 0$ is to be set. The transient current is given (3.3-10), the graph is shown in Fig. 11.

$$I(t) = K \cdot e^{-\int \frac{R}{L} \cdot dt} + \left[\int \left(\frac{U_G}{L} \cdot e^{\int \frac{R}{L} \cdot dt} \right) dt \right] \cdot e^{-\int \frac{R}{L} \cdot dt} \qquad (3.3\text{-}8)$$

$$I(t) = \frac{U_G(t)}{R} \cdot \left(1 - e^{-\frac{R}{L} \cdot t} \right) \qquad (3.3\text{-}9)$$

In case of a sinusoidal external voltage $U_G(t) = U_0.\sin(wt)$ equation (3.3-8) modifies to (3.3-10). The transient current is given (3.3-11). Again from physical reasons as initial condition $I(t=0) = 0$ is to be set. This specifies the constant of integration K as given in (3.3-12). The graph is shown in Fig. 12.

$$I(t) = K \cdot e^{-\int \frac{R}{L} \cdot dt} + \frac{U_0}{L}\left[\int \left(\sin(\omega t) \cdot e^{\int \frac{R}{L} \cdot dt} \right) dt \right] \cdot e^{-\int \frac{R}{L} \cdot dt} \qquad (3.3\text{-}10)$$

$$I(t) = K \cdot e^{-\frac{R}{L}t} + \frac{U_0}{R} \cdot \frac{\sin(\omega t) - \omega \cdot L/R \cdot \cos(\omega t)}{1 + (\omega \cdot L/R)^2} \qquad (3.3\text{-}11)$$

$$K = \frac{U_0 \cdot \omega \cdot L/R}{1 + (\omega \cdot L/R)^2} \qquad (3.3\text{-}12)$$

f ' = a.f(t) ; a = const.

a = -R/L	-0,1
R	10
L	100
U_B	8

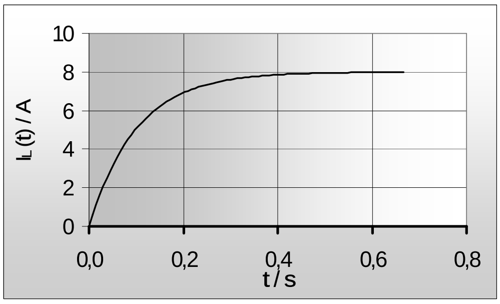


Fig. 11: R-L series under DC conditions when closing the switch.

f' = a.f(t) + g₀.sin(wt.); a = const.

a:=-L/R			
R	*1000*		
L	*1000*	*a*	*-1,00000*
f	*1*	*↑*	*6,28*
g_0:=U_0	*10,00*		
K	*1,55223*		

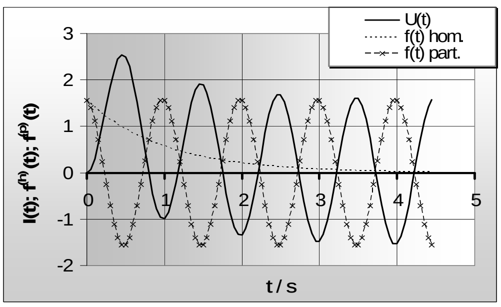


Fig. 12: Transient current through a magnetic coil with a serial resistor under sinusoidal AC conditions after switching on.

3.3.2 To switch off a magnetic coil

To switch off inductivity causes an abrupt decrease of the current in the electric circuit. This means theoretically: $dI(t)/dt \uparrow^{\infty}$ (Fig. 13a). This will not be realised in nature. During a finite circuit time the complete energy stored in the magnetic field must be removed by a current flowing over the contacts of the switch during the opening process. For the short circuit time this energy is much higher than the “electron work of emission” of the contact material of the switch. Therefore an electric current between the opening contacts occurs. It ionises any gas between the contacts and creates a plasma hose with outstanding high temperatures. These energetic conditions and temperatures destroy any switch. Therefore technical solutions are realised to avoid this as shown in Fig. 13 b and c. Another technical solution is to connect a capacitor parallel to the R-C serial as shown in Fig. 14. When opening the switch the capacitor stores the energy in a specific way. To understand the interaction between the resistor, the coil and the capacitor in detail Kirchhoff´s law (3.2-6) is used. Notice that no external voltage $U_G(t)$ is impressed when opening the switch. This leads to equation (3.3-13).

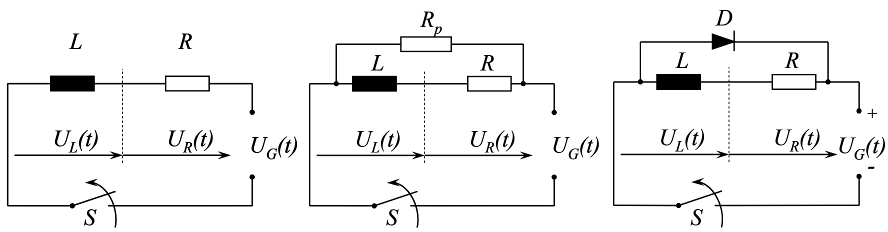


Fig. 13: Connection scheme: to switch off a magnetic coil: a) without anything, b) with a parallel resistor R_p, c) with a diode operated in reverse biasing.

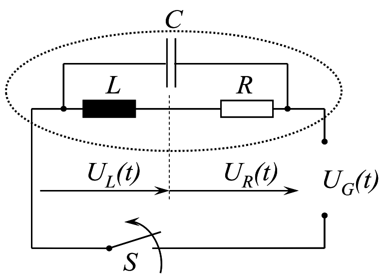


Fig. 14: Connection scheme: to switch off a magnetic coil with a parallel capacitor

$$\sum_{i=1}^{n} U_i(t) \;=\; U_L(t) + U_R(t) + U_C(t) \;=\; 0 \qquad (3.3\text{-}13)$$

With use of (3.2-3), (3.2-5) and (3.3-5) equation (3.3-14) occurs.

$$L \cdot \frac{dI(t)}{dt} + R \cdot I(t) + \frac{1}{C}\int I(t) \cdot dt \;=\; 0 \qquad (3.3\text{-}14)$$

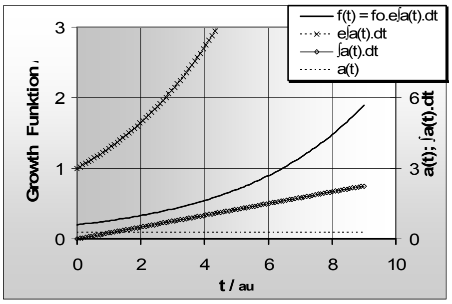
After division by L and differentiation with respect to t a new type of differential equation occurs. It is from 2^{nd} order in t and denotes a completely different behaviour of the described system as discussed in this topic. The acting principle as mentioned at the beginning: “the change of the effect is proportional to the actual magnitude of the effect” is not valid for the acting system as shown in Fig. 14 and mathematically described by equation (3.3-15). To connect a capacitor parallel to the R-L- serial creates a new acting system which shows an oscillation. This will be discussed in detail separately. It should be noticed that this new type of differential equation is inhomogeneous too when a driving force is acting. From physical reasons the coefficients might be variable. But to understand the principals of oscillation and to describe them with more or the less easy mathematical methods later considerations will restrict to constant coefficients.

$$\frac{dI^2(t)}{dt^2} + \frac{R}{L} \cdot \frac{dI(t)}{dt} + \frac{1}{LC} \cdot I(t) \;=\; 0 \qquad (3.3\text{-}15)$$

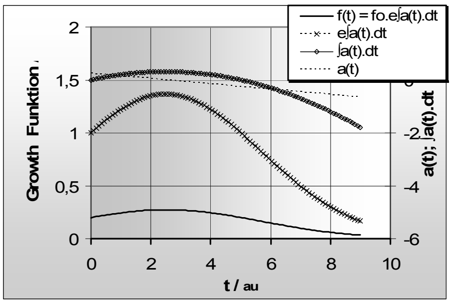
3.4 Effects with different growth factors *a(t)*

Here specific assumptions for growth factors *a(t)* are made. Linear increasing and decreasing behaviour and periodic growth factors are investigated. But the considered effects are assumed not to be dependent from any inhomogeneity. No analytical solution is given because for some of the describing integrals no antiderivative may be found analytically. Therefore only a numerical approach was done using common software / spreadsheet (*Excel*®). For the specific problem the assumptions are figured in the tables left of any plot in Fig. 15.

$f' = a \, . \, f(t);\ a = (k.t+d) + \sin(\omega t + \varphi)$	
k	0,00
d	0,25
ν /Hz	0,00
ω	0,00
φ	0,00
g_0	2,00

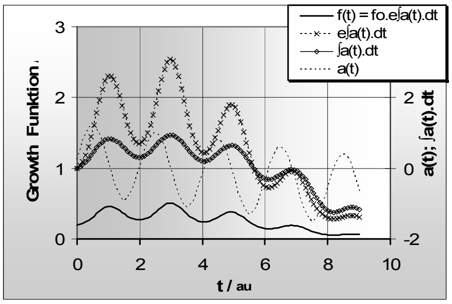


$f' = a \, . \, f(t);\ a = (k.t+d) + \sin(\omega t + \varphi)$	
k	-0,10
d	0,25
ν /Hz	0,00
ω	0,00
φ	0,00
g_0	2,00

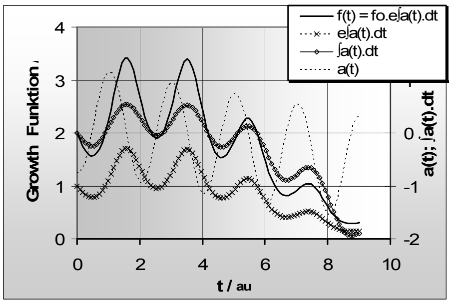


W. Hohenauer:

$f' = a \,.\, f(t);\ a = (k.t+d) + \sin(\omega t + \varphi)$	
k	-0,10
d	0,25
ν /Hz	0,50
ω	3,14
φ	0,00
g_0	2,00



$f' = a \,.\, f(t);\ a = (k.t+d) + \sin(\omega t + \varphi)$	
k	-0,10
d	0,25
ν /Hz	0,50
ω	3,14
φ	-8,00
g_0	2,00



$f' = a \,.\, f(t);\ a = (k.t+d) + \sin(\omega t + \varphi)$	
k	0,00
d	0,00
ν /Hz	0,50
ω	3,14
φ	-8,00
g_0	2,00

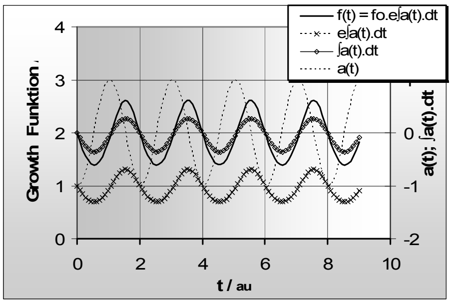


Fig. 15: Examples for growth effects based on different growth factors a(t).